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SUPERCOMPACT SPACES

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SUPERCOMPACT SPACES submitted by MURRAY GORDON BELL in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

TO WANDA

ABSTRACT

A space X is supercompact if X possesses an open subbase $\mathcal{S}$ such that every open cover of X from $\mathcal{S}$ has a two subcover. This concept was first introduced by the Dutch mathematician, J. de Groot. It had been conjectured that all compact Hausdorff spaces are supercompact. This problem has been the motivating force behind this thesis. We show that if X is non-pseudocompact, then βX is non-supercompact. Furthermore, we extend this result to encompass the case where two is replaced by a larger integer in the definition of supercompact. These results rely heavily on combinatorial properties of subsets of ω (the first infinite ordinal). Consequently, considerable attention is devoted to the existence of certain transversals of collections of subsets of ω.

Supercompact spaces have rigid cellular requirements. We investigate them and show that if γX is a super-compactification of X then the cellularity of $\gamma X - X$ cannot exceed the weight of the space X.

The idea of breadth in topological spaces is introduced. In particular, a space X has breadth two if X possesses an open subbase such that the union of three members of $\mathcal{S}$ is actually the union of two of the three members. Hence, if a compact space has breadth two, then every closed subspace is supercompact. In general, supercompactness is not a closed hereditary property. Using a combinatorial proposition in lattice theory, which we develop, we show that every one-dimensional separable metric space has breadth two.

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Murray G. Bell (1977)

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CHAPTER □

Introduction

This thesis will be concerned with special kinds of subbases for topological spaces. A collection $\mathcal{S}$ of open subsets of a topological space X is called an *open subbase* if the open sets of X are precisely the arbitrary unions of finite intersections of $\mathcal{S}$. Dually, a collection $\mathcal{S}$ of closed subsets of a topological space X is called a *closed subbase* if the closed sets of X are precisely the arbitrary intersections of finite unions of $\mathcal{S}$. Throughout, we shift from one to the other, whichever is best suited to the purpose at hand.

Traditionally, subbases have played a small but key rôle in topology. Examples are the definition of the product topology and the original proof of Tychonov's theorem for compact spaces. With regard to the latter, Alexander's subbase theorem was the crucial ingredient. It reads as follows: A topological space X is compact if and only if X possesses an open subbase such that any cover of X by members of this subbase has a finite subcover. The main topological concept motivating this thesis was derived from this theorem and first introduced by the Dutch mathematician J. de Groot [11].

Definition: A space X is *supercompact* if X possesses an open subbase such that any cover of X by members of this subbase has a subcover of two members.

Observing the abundance of supercompact spaces, de Groot had wondered whether all compact Hausdorff spaces were supercompact. This

problem is solved in the negative. Part of this research appears in M. Bell [2] and [3].

Chapter I is concerned with finding certain transversals on collections of subsets of the natural numbers. This finds application in the subbase problems of Chapter III. Chapter II is concerned with a combinatorial result on the breadth of a 0-1 distributive lattice. This is applied in Chapter IV where a new topological concept, that of breadth, is introduced. It is shown that a one-dimensional separable metric space possesses a very special kind of subbase. This concept is an offshoot of supercompactness.

Our set-theoretical notation is standard. For a set X, $|X|$ denotes the cardinality of X and $\mathcal{P}(X)$ is the set of all subsets of X. For a collection $\mathcal{S}$ of sets, $\cup\mathcal{S} = \{x: x \in S \text{ for some } S \in \mathcal{S}\}$ and $\cap\mathcal{S} = \{x: x \in S \text{ for every } S \in \mathcal{S}\}$. The first infinite ordinal is ω, the first uncountable ordinal is ω_1 and c is the cardinality of the real numbers. If N is a positive integer, then $[X]^N$ denotes the set of all subsets of X of cardinality N and $[X]^{<\omega}$ denotes the set of all finite subsets of X.

The lattice theory used in Chapter II is elementary. All the reader need know is the definition of a 0-1 distributive lattice. For this, see "Lattice Theory" by G. Birkhoff [4].

A Tychonov space is a completely regular Hausdorff space. For such spaces X, βX denotes the Stone-Cêch compactification of X. If f is a continuous real-valued function on X, then $\{x \in X: f(x) = 0\}$ is called a *zero-set* of X and $\{x \in X: f(x) \neq 0\}$ is called a *cozero-set* of X. $\mathcal{Z}(X)$ denotes the set of all zero-sets of X. βX is characterized as being that compactification of X such that

(1) $\{C\ell_{\beta X}Z\colon Z \in Z(X)\}$ is a base for the closed sets of βX.

(2) $\{Z_1, Z_2\} \subseteq Z(X)$ implies $C\ell_{\beta X}(Z_1 \cap Z_2) = C\ell_{\beta X}Z_1 \cap C\ell_{\beta X}Z_2$.

A space X is *pseudocompact* if there are no continuous unbounded real-valued functions on X. A space X is *countably compact* if every infinite subset has a cluster point. Countably compact spaces are pseudocompact. Y is a *neighbourhood retract* of X if there exists an open subspace U of X containing Y and a continuous map $r\colon U \to Y$ such that $r(y) = y$ for all $y \in Y$. I, R, Q, N and 2 denote the closed unit interval, the reals, the rationals, the naturals and the two point discrete space respectively. A good reference for this paragraph and the other standard topological concepts used in this thesis is the excellent book of S. Willard, "General Topology" [37].

The cardinal functions used are as follows. The *weight* of a space, $w(X)$, is the least cardinal of an open base for X. The *cellularity* of a space, $c(X)$, is the supremum of $|G|$, where G is a disjoint collection of open sets of X. The *density* of a space, $d(X)$, is the least cardinal of a dense subspace of X. The *spread* of a space, $s(X)$, is the supremum of $|D|$, where D is a discrete subspace of X. The reader is referred to I. Juhász's book, "Cardinal Functions in Topology" [15].

The dimension theory of Chapter IV takes place in the realm of separable metric spaces. In this realm, all three of the standard dimension functions ind, Ind and dim are equal. This was proven by W. Hurewicz and H. Wallman [14]. We shall write $\dim X$ for the dimension of such spaces X. For $A \subseteq X$, the boundary of A in X is $C\ell_X A \cap C\ell_X(X-A)$. It is denoted by $Bd_X A$, or when there is no confusion,

by Bd A. The definition of dimension that we use is given inductively as follows:

(1) $\dim \emptyset = -1$

(2) $\dim X \leq n$ if for every closed subset C of X and for every open subset O of X with $C \subseteq O$, there exists an open subset V of X with $C \subseteq V \subseteq O$ and $\dim(\mathrm{Bd}\, V) \leq n-1$.

CHAPTER I

Transversals

I.1. Introduction. Our motivation for this study of transversals originates from questions on the space $\beta\omega$ of ultrafilters of ω. We recall the pertinent details. The underlying set for $\beta\omega$ is the collection of all ultrafilters on ω. For $A \subseteq \omega$, let $\bar{A} = \{p \in \beta\omega: A \in p\}$. Then $\{\bar{A}: A \subseteq \omega\}$ is taken as a base for the closed sets of $\beta\omega$. Since $\beta\omega - \bar{A} = \overline{\omega - A}$, this is also a base for the open sets of $\beta\omega$. $\beta\omega$ becomes a compact, Hausdorff, 0-dimensional space. A collection $\mathcal{S}$ of closed sets of $\beta\omega$ which is closed under finite intersections is a subbase iff for each $A \subseteq \omega$, there exists a finite subcollection $\{S_1,\cdots,S_n\}$ of $\mathcal{S}$ with $\bar{A} = \cup\{S_i: 1 \leq i \leq n\}$. This follows from compactness and 0-dimensionality. Consider the trace of $\mathcal{S}$ on ω, i.e. $\mathcal{G} = \{S \cap \omega: S \in \mathcal{S}\}$. Let $A \subseteq \omega$. Then, $\bar{A} = \cup\{S_i: 1 \leq i \leq n\}$ implies $A = \cup\{S_i \cap \omega: 1 \leq i \leq n\}$. Thus $\mathcal{G}$ "generates" $\mathcal{P}(\omega)$ under finite unions. More formally, $\mathcal{G} \subseteq \mathcal{P}(\omega)$ is called a *generating set* for $\mathcal{P}(\omega)$ if each subset of ω is a finite union of members of $\mathcal{G}$. The richness of $\mathcal{P}(\omega)$ forces any such generating set to contain certain finite subsystems. If, as in the case above, the generating set "comes" from a closed subbase $\mathcal{S}$, then these finite subsystems can be lifted into $\mathcal{S}$. For example, in Chapter III it will be shown that $\mathcal{S}$ must contain three elements whose total intersection is empty but each pair has nonempty intersection. This is reasonable, but not at all obvious.

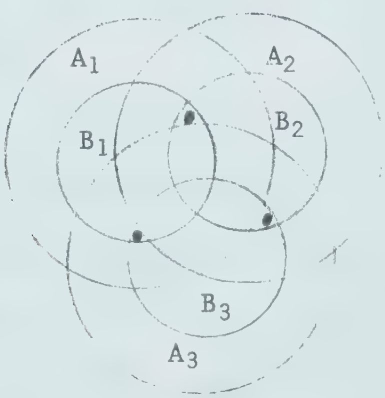
The main tool employed in this project is the idea of a transversal.

Several authors have given different definitions of a transversal. For an excellent source on transversals of families of finite sets, the reader is referred to L. Mirsky's book, "Transversal Theory" [24]. For an example of work being done on transversals in infinitary combinatorics, see the paper of E. Milner [23].

By an N *fold intersection* is meant an intersection of N distinct sets. It is helpful to make the following definition.

I.2. Definition. Let N be a positive integer. Let $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ and $\mathcal{B} = \{B_\gamma : \gamma \in \Gamma\}$ be two collections of sets such that for each $\gamma \in \Gamma$, $B_\gamma \subseteq A_\gamma$. Then T is called an N *transversal on* $\mathcal{B}/\mathcal{A}$ if $T \subseteq \cup\mathcal{B}$, T intersects all N fold intersections from $\mathcal{B}$ in a singleton and T is disjoint from all $N+1$ fold intersections from $\mathcal{A}$. If for all $\gamma \in \Gamma$, $B_\gamma = A_\gamma$, then we simply say that T is an N transversal on $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are as indicated in the following diagram and T consists of the three points indicated, then T is a 2 transversal on $\mathcal{B}/\mathcal{A}$.



$\mathcal{A} = \{A_1, A_2, A_3\}$

$\mathcal{B} = \{B_1, B_2, B_3\}$

$T = \{\cdot, \cdot, \cdot\}$

Our interest lies in two directions. First, the existence of an uncountable collection of subsets of ω satisfying a prescribed property and second, the existence of an N transversal, for some N, on a countable subsystem. Note that one cannot have an N transversal on an uncountable collection of subsets of ω.

I.3. Proposition. Let N be a positive integer. There exists $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\omega)$ with N fold intersections infinite and $N+1$ fold intersections finite.

Proof: When $N = 1$, this is a well-known result in W. Sierpiński [30], page 81. In this case, such a collection is called "almost disjoint". So assume $N > 1$. We shall construct the A_α's inductively. Independently, we construct A_i for $i < \omega$. This is necessary because there are finite collections which are maximal with respect to the properties stated in the proposition.

Enumerate $[\omega]^N$ as $\{H_i : i < \omega\}$. Let $\omega = \cup\{S_i : i < \omega\}$ be a partition of ω into infinitely many infinite subsets. For $i < \omega$, define $A_i = \cup\{S_j : i \in H_j\}$. If $H_n \in [\omega]^N$, then $\cap\{A_i : i \in H_n\} = S_n$ and if $F \in [\omega]^{N+1}$, then $\cap\{A_i : i \in F\} = \emptyset$.

Continuing on, assume that for $\alpha < \beta < \omega_1$ and $\beta > \omega$, A_α has been defined such that

(1) $H \in [\beta]^N$ implies $\cap\{A_\alpha : \alpha \in H\}$ is infinite.

(2) $F \in [\beta]^{N+1}$ implies $\cap\{A_\alpha : \alpha \in F\}$ is finite.

Since $\{A_\alpha : \alpha < \beta\}$ is countable, re-well-order this set as an ω-sequence, $\{B_i : i < \omega\}$. Enumerate $[\omega]^{N-1}$ as $\{D_i : i < \omega\}$. For each $m \notin D_i$, choose $c_{im} \in (B_m \cap \cap\{B_j : j \in D_i\}) - (\cup\{B_k : k < m,\ k \notin D_i\} \cup \{c_{ip} : p < m\})$. Let $C_i = \{c_{im} : m \notin D_i,\ m > i\}$. Then C_i has the following properties:

(a) C_i is an infinite subset of $\cap\{B_j: j \in D_i\}$.

(b) For each $H \in [\omega]^N$ and for each i, $C_i \cap \cap\{B_j: j \in H\}$ is finite.

(c) For each $H \in [\omega]^N$, there exists an i_H such that $i > i_H$ implies $C_i \cap \cap\{B_j: j \in H\} = \emptyset$. To see this, choose i_H such that $\{B_j: j \in H\} \subseteq \{B_i: i < i_H\}$.

Hence, defining $A_\beta = \cup\{C_i: i < \omega\}$ completes the inductive step and $\{A_\alpha: \alpha < \omega_1\}$ is as desired. □

We now supply the companion to Proposition I.3.

I.4. Proposition. Let N be a positive integer. Let $\mathcal{A} = \{A_\alpha: \alpha < \omega_1\}$ and $\mathcal{B} = \{B_\alpha: \alpha < \omega_1\}$ be two families of subsets of ω such that

(a) For all $\alpha < \omega_1$, $B_\alpha \subseteq A_\alpha$.

(b) N fold intersections from $\mathcal{B}$ are infinite.

(c) $N+1$ fold intersections from $\mathcal{A}$ are finite.

Then there exist $\{\alpha_i: i < \omega\} \subseteq \omega_1$ and a $T \subseteq \omega$ with T an N transversal on $\{B_{\alpha_i}: i < \omega\}/\{A_{\alpha_i}: i < \omega\}$.

Proof: For $N = 1$, proceed as follows: Consider $\{A_o \cap A_\alpha: 0 < \alpha < \omega_1\}$. There exists an uncountable subset M_o of $\omega_1 - \{0\}$ such that for every $\beta \neq \gamma$ in M_o, $A_o \cap A_\beta = A_o \cap A_\gamma$. Let $\alpha_o = 0$.

For $N > 1$, let $\alpha_o = 0$ and $M_o = \omega_1 - \{\alpha_o\}$. Thus, assume that we have constructed $\{\alpha_o,\cdots,\alpha_m\}$ and $\{M_o,\cdots,M_m\}$ such that

(1) $0 \leq i \leq m$ implies $\alpha_i \in M_{i-1} - M_i$ $(M_{-1} = \omega_1)$

(2) $M_m \subseteq M_{m-1} \cdots M_o \subseteq M_{-1}$ and $|M_m| = \omega_1$

(3) For $F \in [\{\alpha_o,\cdots,\alpha_m\}]^N$ and $\beta \neq \gamma$ in M_m, $\cap\{A_\alpha: \alpha \in F\} \cap A_\beta = \cap\{A_\alpha: \alpha \in F\} \cap A_\gamma$.

Choose $\alpha_{m+1} \in M_m$. Let $\{F_i : 1 \le i \le r\}$ enumerate the N element subsets of $\{\alpha_o, \cdots, \alpha_{m+1}\}$. There exists an uncountable subset N_1 of $M_m - \{\alpha_{m+1}\}$ such that for every $\beta \neq \gamma$ in N_1, $\cap\{A_\alpha : \alpha \in F_1\} \cap A_\beta = \cap\{A_\alpha : \alpha \in F_1\} \cap A_\gamma$. Now choose an uncountable subset N_2 of N_1 such that for every $\beta \neq \gamma$ in N_2, $\cap\{A_\alpha : \alpha \in F_2\} \cap A_\beta = \cap\{A_\alpha : \alpha \in F_2\} \cap A_\gamma$. Proceed in this fashion to N_r. Let $M_m = N_r$. The inductive step is complete.

The set $\cup\{\cap\{A_\alpha : \alpha \in F\} : F \in [\{\alpha_i : i < \omega\}]^{N+1}\}$ meets every N fold intersection from $\{A_{\alpha_i} : i < \omega\}$ in a finite set. So, for every $H \in [\{\alpha_i : i < \omega\}]^N$ choose $\tau_H \in \cap\{B_\alpha : \alpha \in H\} - \cup\{\cap\{A_\alpha : \alpha \in F\} : F \in [\{\alpha_i : i < \omega\}]^{N+1}\}$. Then $T = \{\tau_H : H \in [\{\alpha_i : i < \omega\}]^N\}$ is an N transversal on $\{B_{\alpha_i} : i < \omega\}/\{A_{\alpha_i} : i < \omega\}$. □

We present two examples related to the just proven proposition.

I.5. <u>Example 1</u>. It is necessary that $\mathcal{A}$ and $\mathcal{B}$ be uncountable collections. Enumerate $[\omega]^N$ as $\{H_i : i < \omega\}$. Let $\{S_i : i < \omega\}$ be a countable partition of ω into infinite sets. Define $A_n = B_n = \cup\{S_i : n \in H_i\} \cup \{1, \cdots, n\}$. If $H_i \in [\omega]^N$, then $\cap\{B_n : n \in H_i\}$ contains S_i, hence is infinite. If $F \in [\omega]^{N+1}$, then $\cap\{A_n : n \in F\} \subseteq \{i : i \le \max F\}$, hence is finite. However, every member of ω is in all but finitely many A_n's. Consequently, no transversal could possibly exist on any infinite subsystem.

<u>Example 2</u>. There need not be an $N+1$ transversal on even an $N+2$ element subsystem. This construction is due to W. Sierpiński [30], page 81. Let S be a subset of the real numbers of cardinality ω_1. For each $x \in S$ define $A_x = B_x = \{2^n(2[nx]+1) : n < \omega\}$, where $[nx]$ denotes the greatest integer in nx. 1-fold intersections of the A_x's

are infinite and 2 fold intersections of the A_x's are finite. Hence, by the proposition, we are guaranteed a 1 transversal on an infinite subsystem. However, as the reader can readily check, whenever x, y, z are three distinct numbers in S, modulo a permutation, $|x-y| < |x-z|$ must obtain, whence $A_x \cap A_z \subseteq A_y$. Thus any set which intersects $A_x \cap A_z$ must intersect $A_x \cap A_y \cap A_z$, therefore no 2 transversal can exist on any 3 element subsystem. Thus we see that Sierpiński's example of an uncountable almost disjoint collection of subsets of ω enjoys the added property that the intersection of three of them is actually the intersection of two of the three (two times).

We say that a collection $\mathcal{C}$ of sets *satisfies* P if for all finite $\mathcal{F} \subseteq \mathcal{C}$ and for all $X \in \mathcal{C} - \mathcal{F}$, $X - \cup\mathcal{F}$ is infinite.

Let us prove the following variation of a result due to D.A. Martin and R.M. Solovay [16]. It is a useful tool in what follows.

I.6. Proposition. Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of ω. Assume $|\mathcal{A} \cup \mathcal{B}| = \omega$ and $\mathcal{A} \cup \mathcal{B}$ satisfies P. Then there exists an infinite $C \subseteq \omega$ such that for all $A \in \mathcal{A}$, $C \cap A$ is finite and for all $B \in \mathcal{B}$, $C \cap B$ is infinite and $\mathcal{A} \cup \mathcal{B} \cup \{C\}$ satisfies P.

Proof: Enumerate $\mathcal{A} \cup \mathcal{B}$ as $\{D_n : n < \omega\}$.

Pick $x_{nm} \in D_n - \left(\cup\{D_j : j < m,\ D_j \neq D_n\} \cup \{x_{nj} : j < m\}\right)$.

Let $C = \{x_{nm} : m > n,\ m \text{ even and } D_n \in \mathcal{B}\} \cup \{x_{nn} : n < \omega\}$. □

Two examples are now given to demonstrate the peculiar behaviour that certain uncountable collections of subsets of ω display. They originally caused the author some concern in his investigations.

I.7. Example 1. There exists an uncountable collection $\{C_\alpha : \alpha < \omega_1\}$ of subsets of ω with pairwise intersections infinite and a decomposition of each C_α into two sets $C_\alpha = C_{\alpha 0} \cup C_{\alpha 1}$ such that no three of $\{C_{\alpha_i} : \alpha < \omega_1,\ i \in \{0,1\}\}$ have pairwise intersections infinite.

The reader should observe that upon replacing pairwise and two by finite and finitely many in the above, an application of Zorn's Lemma would yield an uncountable collection of $C_{\alpha i}$'s having finite intersections infinite.

We first construct two almost disjoint collections $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ such that for all (α,β) in $\omega_1 \times \omega_1$, $A_\alpha \cap B_\beta$ is infinite. Let us first construct them up to ω. Partition ω into $\omega \times \omega$ disjoint infinite sets $\omega = \cup\{S_{ij} : (i,j) \in \omega \times \omega\}$. Define $A_n = \cup\{S_{nj} : j < \omega\}$ and $B_n = \cup\{S_{in} : i < \omega\}$. Proceeding inductively, assume that for $\omega < \beta < \omega_1$, $\{A_\alpha : \alpha < \beta\}$ and $\{B_\alpha : \alpha < \beta\}$ have been constructed such that

(1) $\gamma < \alpha < \beta$ implies $A_\gamma \cap A_\alpha$ is finite and $B_\gamma \cap B_\alpha$ is finite.

(2) $(\gamma,\alpha) \in \beta \times \beta$ implies $A_\gamma \cap B_\alpha$ is infinite.

(3) $\{A_\alpha : \alpha < \beta\} \cup \{B_\alpha : \alpha < \beta\}$ satisfies P.

Now we construct A_β and B_β. One application of Proposition I.6 yields an A_β which intersects all the A_α's in a finite set and which intersects all the B_α's in an infinite set and is such that $\{A_\alpha : \alpha \leq \beta\} \cup \{B_\alpha : \alpha < \beta\}$ satisfies P. A second application of Proposition I.6 yields the desired B_β, along with $\{A_\alpha : \alpha \leq \beta\} \cup \{B_\alpha : \alpha \leq \beta\}$ satisfying P. This completes the inductive step.

Define $C_{\alpha 0} = A_\alpha \cap \cup\{B_\gamma : \alpha < \gamma < \omega_1\}$, $C_{\alpha 1} = B_\alpha \cap \cup\{A_\gamma : \gamma < \alpha < \omega_1\}$ and $C_\alpha = C_{\alpha 0} \cup C_{\alpha 1}$. Then C_α, $C_{\alpha 0}$ and $C_{\alpha 1}$ are as required. If $\alpha < \beta$, then $A_\alpha \cap B_\beta \subseteq C_\alpha \cap C_\beta$, hence pairwise intersections of the

C_α's are infinite. However, for any three $C_{\alpha i}$'s chosen, two come from an almost disjoint collection.

A question, posed by the author, led to the following elegant example constructed by George Tokarsky (personal communication).

Example 2. There exists an uncountable collection of infinite subsets of ω which contains neither an uncountable almost disjoint subcollection nor an uncountable subcollection with pairwise intersections infinite.

Let D denote the diagonal of $\omega_1 \times \omega_1$. W. Sierpiński [31] has shown that there exists an $f: (\omega_1 \times \omega_1) - D \to \{0,1\}$ such that

(a) $f(\alpha,\beta) = f(\beta,\alpha)$

and

(b) for each uncountable $S \subseteq \omega_1$, $f|\big((S \times S) - D\big)$ is onto $\{0,1\}$.

To see this, let $\{r_\alpha : \alpha < \omega_1\}$ be a well-ordering of a subset of the reals of cardinality ω_1. Define f as follows: $f(\alpha,\beta) = 1$ if $(\alpha < \beta$ and $r_\alpha < r_\beta)$ or $(\beta < \alpha$ and $r_\beta < r_\alpha)$. $f(\alpha,\beta) = 0$ otherwise. (b) is satisfied since the reals do not contain an uncountable well-ordered sequence under the usual ordering.

We shall construct $\{A_\alpha : \alpha < \omega_1\}$ such that $A_\alpha \cap A_\beta$ is finite iff $f(\alpha,\beta) = 0$. Clearly, this will then be our required collection. First, alter $\{r_\alpha : \alpha < \omega_1\}$ if need be, so that $f(n,m) = 0$ for $n \neq m$, $n < \omega$, $m < \omega$. This can be accomplished by letting $\{r_n : n < \omega\}$ be a strictly decreasing sequence of real numbers. Let $\{A_n : n < \omega\}$ be a partition of ω into infinite sets. Assume A_α have been constructed for $\alpha < \beta < \omega_1$ where $\beta > \omega$ such that

(1) $\gamma < \alpha < \beta$ implies $A_\gamma \cap A_\alpha$ is finite iff $f(\gamma,\alpha) = 0$.

(2) $\{A_\alpha : \alpha < \beta\}$ satisfies P.

Let $\mathcal{A} = \{A_\alpha : f(\alpha,\beta) = 0\}$ and $\mathcal{B} = \{A_\alpha : f(\alpha,\beta) = 1\}$. Proposition I.6 yields an A_β which intersects members of $\mathcal{A}$ in a finite set and members of $\mathcal{B}$ in an infinite set and is such that $\mathcal{A} \cup \mathcal{B} \cup \{A_\beta\}$ satisfies P. This completes the inductive step. $\{A_\alpha : \alpha < \omega_1\}$ is as desired.

Let us investigate a stronger property than P. A collection of sets $\mathcal{C}$ is called an *independent family* if for each pair of disjoint finite subsets of $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{G}$, $\cap\mathcal{F} - \cup\mathcal{G}$ is infinite. The existence of an independent family of cardinality c of subsets of ω was first proved by Fichtenholz and Kantorovitch [9]. F. Hausdorff [13] shortly after came up with an easier proof. The following simple topological proof is implied in Engelking [8] (163-164).

Consider 2^c, the cartesian product of c copies of the two point discrete space. Let $\{d_n : n < \omega\}$ be a countable dense subset of 2^c. For $0 \leq \alpha < c$ let p_α be the projection map onto the α'th coordinate space. Define $D_\alpha = \{n : p_\alpha(d_n) = 1\}$. Then $\{D_\alpha : \alpha < c\}$ is an independent family of subsets of ω.

I.8. <u>Definition</u>. Let $\{A_\gamma : \gamma \in \Gamma\}$ and $\{B_\gamma : \gamma \in \Gamma\}$ be two collections of sets such that for all $\gamma \in \Gamma$, $B_\gamma \subseteq A_\gamma$. Then $\{B_\gamma : \gamma \in \Gamma\}$ is *independent over* $\{A_\gamma : \gamma \in \Gamma\}$ if for each pair of disjoint finite subsets F and G of Γ, $\cap\{B_\gamma : \gamma \in F\} - \cup\{A_\gamma : \gamma \in G\}$ is infinite.

I.9. <u>Lemma</u>. Let $\{A_\alpha : \alpha < \omega_1\}$ be an uncountable independent family of subsets of ω. Assume each A_α is a union of n sets $A_{\alpha 1}, \cdots, A_{\alpha n}$. Then there exists an uncountable subset M of ω_1 and for each $\alpha \in M$

an n_α with $1 \le n_\alpha \le n$ such that $\{A_{\alpha n_\alpha} : \alpha \in M\}$ is independent over $\{A_\alpha : \alpha \in M\}$.

Proof: A somewhat stronger statement will be proven. Given $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ such that

(1) $B_\alpha \subseteq A_\alpha$ and $B_\alpha = \cup\{B_{\alpha i} : 1 \le i \le n\}$

(2) $\{B_\alpha : \alpha < \omega_1\}$ is independent over $\{A_\alpha : \alpha < \omega_1\}$

then there exists an uncountable subset M of ω_1 and for each $\alpha \in M$ an n_α with $1 \le n_\alpha \le n$ such that $\{B_{\alpha n_\alpha} : \alpha \in M\}$ is independent over $\{A_\alpha : \alpha \in M\}$. When $B_\alpha = A_\alpha$ and $B_{\alpha i} = A_{\alpha i}$, we get the lemma.

Induction will be on n. The case $n = 1$ is obvious. So assume the statement is true for n and let $B_\alpha = \cup\{B_{\alpha i} : 1 \le i \le n+1\}$. The $B_{\alpha n_\alpha}$'s are now constructed inductively. Assume we have chosen M_α and N_α for $\alpha < \beta < \omega_1$ such that

(1) $M_\alpha \cup N_\alpha \subseteq \omega_1$, N_α is co-countable in ω_1 and $M_\alpha \cap N_\alpha = \emptyset$.

(2) $\gamma < \alpha$ implies $M_\gamma \subsetneq M_\alpha$ and $N_\alpha \subseteq N_\gamma$.

(3) For all disjoint finite subsets F and G of M_α and all disjoint finite subsets H and K of N_α, $(\cap\{B_{\gamma,n+1} : \gamma \in F\} \cap \cap\{B_\gamma : \gamma \in H\}) - \cup\{A_\gamma : \gamma \in G \cup K\}$ is infinite.

If M_β and N_β can now be constructed such that (1), (2) and (3) hold, then $\{B_{\alpha,n+1} : \alpha \in \cup\{M_\beta : \beta < \omega_1\}$ will be independent over $\{A_\alpha : \alpha \in \cup\{M_\beta : \beta < \omega_1\}\}$. To this end, observe that $\cap\{N_\alpha : \alpha < \beta\}$ is again co-countable. For $\gamma \in \cap\{N_\alpha : \alpha < \beta\}$ define $C_\gamma = \cup\{B_{\gamma i} : 1 \le i \le n\}$.

If there exists an uncountable subset P of $\cap\{N_\alpha : \alpha < \beta\}$ such that $\{C_\gamma : \gamma \in P\}$ is independent over $\{A_\gamma : \gamma \in P\}$ then by our inductive hypothesis for n we shall obtain what we want inside of P. Therefore assume that for each uncountable subset P of $\cap\{N_\alpha : \alpha < \beta\}$ there exist disjoint finite subsets F_P and G_P of P with $\cap\{C_\gamma : \gamma \in F_P\}$ -

$\cup\{A_\gamma : \gamma \in G_{\mathcal{P}}\}$ finite.

Striving for a contradiction, assume that for each $\delta \in \cap\{N_\alpha : \alpha < \beta\}$ and each co-countable subset $\mathcal{P}$ of $\cap\{N_\alpha : \alpha < \beta\}$ there exist disjoint finite subsets F_δ and G_δ of $\cup\{M_\alpha : \alpha < \beta\}$ and disjoint finite subsets H_δ and K_δ of $\mathcal{P}$ with $(B_{\delta\, n+1} \cap \bigcap_{\gamma \in F_\delta} B_{\gamma\, n+1} \cap \bigcap_{\gamma \in H_\delta} B_\gamma) - \cup\{A_\gamma : \gamma \in G_\delta \cup K_\delta\}$ finite. Choose an uncountable subset $\mathcal{R}$ of $\cap\{N_\alpha : \alpha < \beta\}$ and for each $\delta \in \mathcal{R}$ a F_δ, G_δ, H_δ and K_δ as above with $\{H_\delta : \delta \in \mathcal{R}\} \cup \{K_\delta : \delta \in \mathcal{R}\}$ being a mutually disjoint collection and $\mathcal{R} \cap (\cup\{H_\delta : \delta \in \mathcal{R}\} \cup \cup\{K_\delta : \delta \in \mathcal{R}\}) = \emptyset$. $\mathcal{R}$ can be constructed inductively using the preceding assumption. Since there are only countably many pairs of disjoint finite subsets of $\cup\{M_\alpha : \alpha < \beta\}$ it follows that there must be two disjoint finite subsets F and G of $\cup\{M_\alpha : \alpha < \beta\}$ and an uncountable subset $\mathcal{P}$ of $\mathcal{R}$ such that for each $\delta \in \mathcal{P}$, $(B_{\delta\, n+1} \cap \bigcap_{\gamma \in F} B_{\gamma\, n+1} \cap \bigcap_{\gamma\, H_\delta} B_\gamma) - \cup\{A_\gamma : \gamma \in G \cup K_\delta\}$ is finite. Now, for this $\mathcal{P}$, there exist disjoint finite subsets $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ of $\mathcal{P}$ with $\cap\{C_\gamma : \gamma \in F_{\mathcal{P}}\} - \cup\{A_\gamma : \gamma \in G_{\mathcal{P}}\}$ finite. Since $\cap\{B_\gamma : \gamma \in F_{\mathcal{P}}\} \subseteq \cap\{C_\gamma : \gamma \in F_{\mathcal{P}}\} \cup \cup\{B_{\delta\, n+1} : \delta \in F_{\mathcal{P}}\}$ it follows that

$$\left(\bigcap_{\gamma \in F} B_{\gamma\, n+1} \cap \cap\{B_\gamma : \gamma \in F_{\mathcal{P}} \cup \bigcup_{\delta \in F_{\mathcal{P}}} H_\delta\}\right) - \cup\{A_\gamma : \gamma \in G \cup G_{\mathcal{P}} \cup \bigcup_{\delta \in F_{\mathcal{P}}} K_\delta\}$$

is finite. This contradicts (3) since F and G are disjoint finite subsets of some M_α for $\alpha < \beta$ and $F_{\mathcal{P}} \cup \bigcup_{\delta \in F_{\mathcal{P}}} H_\delta$ and $G_{\mathcal{P}} \cup \bigcup_{\delta \in F_{\mathcal{P}}} K_\delta$ are disjoint finite subsets of $\cap\{N_\alpha : \alpha < \beta\} \subseteq N_\alpha$.

Consequently choose $\delta \in \cap\{N_\alpha : \alpha < \beta\}$ and a co-countable subset N_β of $\cap\{N_\alpha : \alpha < \beta\}$ such that for disjoint finite subsets F and G of $\cup\{M_\alpha : \alpha < \beta\}$ and disjoint finite subsets H and K of N_β we have that $(B_{\delta\, n+1} \cap \bigcap_{\gamma \in F} B_{\gamma\, n+1} \cap \bigcap_{\gamma \in H} B_\gamma) - \cup\{A_\gamma : \gamma \in G \cup K\}$ is infinite.

Since $\delta \in \cap\{N_\alpha : \alpha < \beta\}$ it is also true that

$$\Big(\bigcap_{\gamma\in F} B_{\gamma\, n+1} \cap \bigcap_{\gamma\in H} B_\gamma\Big) - \big(A_\delta \cup \cup\{A_\gamma : \gamma \in G \cup K\}\big)$$

is infinite. Hence defining $M_\beta = \cup\{M_\alpha : \alpha < \beta\} \cup \{\delta\}$ we see that M_β and N_β satisfy (1) thru (3). This completes the proof. □

I.10. <u>Lemma</u>. Let N be a positive integer. Let $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ be two families of subsets of ω such that

(1) for each $\alpha < \omega_1$, $B_\alpha \subseteq A_\alpha$

(2) $\{B_\alpha : \alpha < \omega_1\}$ is independent over $\{A_\alpha : \alpha < \omega_1\}$. Then there exist $\{\alpha_i : i < \omega\} \subseteq \omega_1$ and a $T \subseteq \omega$ with T an N transversal on $\{B_{\alpha_i} : i < \omega\}/\{A_{\alpha_i} : i < \omega\}$.

<u>Proof</u>: If $N = 1$ proceed as follows: If for all $y \in B_o$, $|\{\beta < \omega_1 : y \notin A_\beta\}| \le \omega$ then $|\{\beta < \omega_1 : B_o \not\subseteq A_\beta\}| < \omega_1$. Thus there exist infinitely many $\beta > 0$ with $B_o \subseteq A_\beta$. Contradiction. Choose $\tau_o \in B_o$ and $M_o \subseteq \omega_1$ with $|M_o| = \omega_1$ and $\tau_o \notin \cup\{A_\alpha : \alpha \in M_o\}$. Let $\alpha_o = 0$. If $N > 1$ then let $M_o = \omega_1 - \{0\}$ and $\alpha_o = 0$.

Assume we have chosen $\{\alpha_o,\cdots,\alpha_m\}$, $\{M_o,\cdots,M_m\}$ and $\{\tau_H : H \in [\{0,\cdots,m\}]^N\}$ such that

(1) $0 \le i \le m$ implies $\alpha_i \in M_{i-1} - M_i$ $(M_{-1} = \omega_1)$

(2) $M_m \subseteq M_{m-1} \cdots M_o \subseteq M_{-1}$ with $|M_m| = \omega_1$.

(3) $\tau_H \in \bigcap_{i\in H} B_{\alpha_i} - \big(\cup\{A_{\alpha_i} : 0 \le i \le m,\ i \notin H\} \cup \cup\{A_\beta : \beta \in M_{\max H}\}\big)$.

Upon completion of the inductive step $T = \{\tau_H : H \in [\omega]^N\}$ will be an N transversal on $\{B_{\alpha_i} : i < \omega\}/\{A_{\alpha_i} : i < \omega\}$. This is true since for all $H \in [\omega]^N$, $T \cap \cap\{B_{\alpha_i} : i \in H\} = \{\tau_H\}$, hence T intersects all N fold intersections from $\{B_{\alpha_i} : i < \omega\}$ in a singleton. Also, each τ_H is in

exactly N A_{α_i}'s namely for those i's in H. Thus T is disjoint from all $N+1$ fold intersections from $\{A_{\alpha_i} : i < \omega\}$. Clearly, $T \subseteq \cup\{B_{\alpha_i} : i < \omega\}$.

With this end in mind choose $\alpha_{m+1} \in \mathcal{M}_m$. Enumerate $\{H : H \in [\{0,\cdots,m+1\}]^N$ and $m+1 \in H\}$ as $\{H_j : 1 \leq j \leq r\}$. For each j such that $1 \leq j \leq r$ choose an uncountable subset $\mathcal{P}_j$ of $\mathcal{M}_m$ and a $\tau_{H_j} \in \cap\{B_{\alpha_i} : i \in H_j\} - (\cup\{A_{\alpha_i} : 0 \leq i \leq m,\ i \notin H_j\} \cup \cup\{A_\beta : \beta \in \mathcal{P}_j\})$ such that if $1 \leq j < k \leq r$ then $\mathcal{P}_k \subseteq \mathcal{P}_j$. For if this could not be achieved then there would exist a j with $1 \leq j \leq r$ and infinitely many $\beta \notin \{\alpha_i : i \in H_j\}$ such that $\cap\{B_{\alpha_i} : i \in H_j\} \subseteq \cup\{A_{\alpha_i} : 0 \leq i \leq m,\ i \in H_j\} \cup A_\beta$ which would contradict independence. Let $\mathcal{M}_{m+1} = \mathcal{P}_r$. Then $\{\alpha_o,\cdots,\alpha_{m+1}\}$. $\{\mathcal{M}_o,\cdots,\mathcal{M}_{m+1}\}$ and $\{\tau_H : H \in [\{0,\cdots,m+1\}]^N\}$ satisfy (1) thru (3). $\square$

Employing the same artifice as in I.5 Example 1, it is readily seen that it is necessary in the above that the collections be uncountable.

Following P. Alexsandrov [1], two collections $\{A_\gamma : \gamma \in \Gamma\}$ and $\{B_\gamma : \gamma \in \Gamma\}$ are said to be *combinatorially equivalent* if for each finite subset F of Γ, $\cap\{A_\gamma : \gamma \in \Gamma\} = \emptyset$ iff $\cap\{B_\gamma : \gamma \in \Gamma\} = \emptyset$. For each $N \geq 2$ define $\mathcal{D}_N = [\{1,\cdots,N\}]^{N-1}$. $\mathcal{D}_N$ can be described combinatorially as a collection of N sets whose total intersection is empty while all $N-1$ fold intersections of them are nonempty.

The reader is now reminded of the following theorem of F.P. Ramsey [28]. "Let n be a positive integer. If $[\omega]^n = \cup\{W_j : 1 \leq j \leq r\}$ then there exist an infinite $A \subseteq \omega$ and an s with $1 \leq s \leq r$ such that $[A]^n \subseteq W_s$."

I.11. <u>Theorem</u>. Let $N \geq 2$. Assume $\mathcal{G}$ is a generating set of $\mathcal{P}(\omega)$ and g is a particular description of $\mathcal{G}$, i.e. $g : \mathcal{P}(\omega) \to [\mathcal{G}]^{<\omega}$ such that for each $A \subseteq \omega$, $A = \cup g(A)$. Then there exist $\mathcal{H} \in [\mathcal{P}(\omega)]^N$ and

for each $H \in \mathcal{H}$ a $G(H) \in g(H)$ such that

(1) $\cap\mathcal{H} = \emptyset$

(2) $\mathcal{H}' \in [\mathcal{H}]^{N-1}$ implies $\cap\{G(H): H \in \mathcal{H}'\} \neq \emptyset$.

In particular, $\{G(H): H \in \mathcal{H}\}$ is a subcollection of $\mathcal{G}$ combinatorially equivalent to $\mathcal{D}_N$.

Proof: For $N = 2$ choose two disjoint nonempty subsets H and K of ω. Choose $G(H) \in g(H) - \{\emptyset\}$ and $G(K) \in g(K) - \{\emptyset\}$. Let $\mathcal{H} = \{H,K\}$. So assume $N > 2$. Let $\{A_\alpha: \alpha < \omega_1\}$ be an uncountable independent family of subsets of ω. Pick an uncountable subset $\mathcal{M}$ of ω_1 and an $n < \omega$ such that for each $\alpha \in \mathcal{M}$, $|g(A_\alpha)| = n$. For $\alpha \in \mathcal{M}$ let $g(A_\alpha) = \{A_{\alpha 0},\cdots,A_{\alpha n}\}$. Lemma I.9 followed by Lemma I.10 yields $\{\alpha_i: i < \omega\} \subseteq \mathcal{M}$, for each $i < \omega$ an n_i with $1 \leq n_i \leq n$ and a $T \subseteq \omega$ with T an $N-2$ transversal on $\{A_{\alpha_i,n_i}: i < \omega\}/\{A_{\alpha_i}: i < \omega\}$. Moreover, $\{A_{\alpha_i,n_i}: i < \omega\}$ has finite intersections infinite.

Let $g(T) = \{G_1,\cdots,G_r\}$ and $W_j = \{F \in [\omega]^{N-2}: T \cap \cap\{A_{\alpha_i,n_i}: i \in F\} \in G_j\}$. Thus $[\omega]^{N-2} = \cup\{W_j: 1 \leq j \leq r\}$. Ramsey's theorem supplies an infinite $A \subseteq \omega$ and an s with $1 \leq s \leq r$ such that $[A]^{N-2} \subseteq W_s$. Choose $N-1$ distinct elements from A w.l.o.g. let them be $1,\cdots,N-1$. Define $\mathcal{H} = \{T\} \cup \{A_{\alpha_i}: 1 \leq i \leq N-1\}$. Let $G(T) = G_s$ and $G(A_{\alpha_i}) = A_{\alpha_i,n_i}$. Since T is an $N-2$ transversal $\cap\mathcal{H} = \emptyset$. Since $\cap\{G(A_{\alpha_i}): 1 \leq i \leq N-1\} \neq \emptyset$ and $[\{1,\cdots,N-1\}]^{N-2} \subseteq W_s$, all $N-1$ fold intersections of the $G(H)$'s for $H \in \mathcal{H}$ are nonempty. $\square$

This theorem will have topological implications in Chapter III. Set theoretically it says that for any $N \geq 2$ a generating set of $\mathcal{P}(\omega)$ contains a subcollection combinatorially equivalent to $\mathcal{D}_N$. To do this, only the idea of a transversal on $\mathcal{A}$ was needed. It is when we want to realize

these configurations in a closed subbase of $\beta\omega$ that the idea of a transversal on $\mathcal{B}/\mathcal{A}$ proves useful. This will also be extended to βX where X is a non-pseudocompact space.

The following question is left open to the reader. Given a finite collection of sets $\mathcal{F}$ and a generating set $\mathcal{G}$ of $\mathcal{P}(\omega)$ (a closed subbase $\mathcal{S}$ of $\beta\omega$), does $\mathcal{G}$ ($\mathcal{S}$) contain a subcollection combinatorially equivalent to $\mathcal{F}$?

CHAPTER II

Breadth in Lattices

II.1. Introduction. This chapter is devoted to a single proposition in Lattice Theory. It will be applied in Chapter IV but it was thought best to isolate it and prove it in its proper setting.

The following concept is the foundation upon which Chapter IV is built.

II.2. Definition. (Garrett Birkhoff [4]). Let L be a lattice. The *breadth of* L, denoted by $b(L)$, is the smallest positive integer (if one exists) such that any join $x_1 \vee x_2 \cdots \vee x_{b+1}$ is always a join of b of the x_i.

It is convenient to extend this notion to an arbitrary subset A of L. The breadth of A, $b(A)$, is the smallest positive integer b (if one exists) such that any join $a_1 \vee a_2 \cdots \vee a_{b+1}$ (with $a_i \in A$) is always a join of b of the a_i. It is clear that a subset A has breadth $\leq b$ iff every subset of A with $b+1$ elements has breadth $\leq b$.

II.3. Proposition. Let L be a 0-1 distributive lattice. Let $\{x_1,\cdots,x_n\} \subseteq L$ with $b\{x_1,\cdots,x_n\} \leq 2$. Assume that for each pair $j \neq k$ there are given elements x_{jk}, x_{kj} such that

(1) $x_{jk} \leq x_j$, $x_{kj} \leq x_k$, $x_{jk} \vee x_k = x_{kj} \vee x_j = x_j \vee x_k$

(2) $x_{jk} \wedge x_{kj} = 0$

(3) for distinct i,j,k $x_i \leq x_{jk} \vee x_{kj} \vee x_{ij} \vee x_{ik}$

(4) a) $i < j < k$ implies $x_i \le x_{ij} \vee x_{jk}$

b) $i < j < k < \ell$ implies $x_k \le x_{ik} \vee x_{ki} \vee x_{kj} \vee x_{j\ell}$.

Let $\mathcal{F} = \{f\colon f$ is a mapping of $[\{1,\cdots,n\}]^2$ into $\{x_{jk}\colon 1 \le j \le n$, $1 \le k \le n$ and $j \ne k\}$ and $f(\{jk\}) \in \{x_{jk}, x_{kj}\}\}$. For each $f \in \mathcal{F}$, let $A_f = \bigvee\{f(F)\colon F \in [\{1,\cdots,n\}]^2\}$. Let $x_{n+1} \in L$. Then $x_{n+1} = \bigwedge_{f \in \mathcal{F}} (x_{n+1} \vee A_f)$ and $b\left(\{x_i\colon 1 \le i \le n\} \cup \{x_{n+1} \vee A_f\colon f \in \mathcal{F}\}\right) \le 2$.

Proof: For brevity we write j in place of x_j and jk in place of x_{jk}. We use the following frequently without mention throughout the proof.

- $jk \vee k = j \vee k$ and $jk \le j$
- for distinct i,j,k, $i \le jk \vee kj \vee ij \vee ik$
- for $f \in \mathcal{F}$, $jk \nleq A_f$ implies $kj \le A_f$.

The proof is divided into four parts of progressive complications. That $n + 1 = \bigwedge_{f \in \mathcal{F}} \left((n+1) \vee A_f\right)$ follows from condition (2) and the distributive law.

Part 1. If $\{a,b,c\} \subseteq \{1,\cdots,n\}$ then $b\{a,b,c\} \le 2$ by assumption.

Part 2. If $\{a,b\} \subseteq \{1,\cdots,n\}$ and $C \in \{A_f\colon f \in \mathcal{F}\}$ then either $ab \le C$ or $ba \le C$, consequently either $a \le b \vee C$ or $b \le a \vee C$. Hence $b\{a,b,(n+1) \vee C\} \le 2$.

Part 3. Let $a \in \{1,\cdots,n\}$ and $\{C,E\} \subseteq \{A_f\colon f \in \mathcal{F}\}$. Striving for a contradiction, assume $b\{a,(n+1) \vee C, (n+1) \vee E\} = 3$. Thus there must be $cd \le C$ and $ef \le E$ with $cd \nleq a \vee E$, $ef \nleq a \vee C$ and $a \nleq C \vee E$. This implies that $dc \le E$ and $fe \le C$. Note that a,c,d,e,f may not all be distinct. Certainly $a \notin \{c,e\}$. If $ea \le C$ then $a \vee C \ge e$. Therefore $ae \le C$. Similarly $ac \le E$.

For the moment assume $a = f$. If $c = e$ then $a \vee E \ge c$. Therefore $c \ne e$. If $ce \le E$ then $a \vee E \ge a \vee ea \vee ce \ge c$. Therefore

$ec \le E$. If $d = e$ then $C \vee E \ge cd \vee dc \vee ac \vee ad \ge a$. Therefore $d \ne e$. If $ed \le E$ then $C \vee E \ge (cd \vee dc \vee ec \vee ed) \vee ae \ge a$. Therefore $de \le E$. If $a = d$ then $C \vee E \ge (ac \vee ca \vee ea \vee ec) \vee ae \ge a$. Therefore $a \ne d$. If $da \le E$ then $C \vee E \ge (ae \vee ea \vee da \vee de) \vee cd \vee ac \ge a$. Therefore $ad \le E$. Hence $C \vee E \ge cd \vee dc \vee ac \vee ad \ge a$. We conclude that $a \ne f$.

If $af \le E$ then $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. Therefore $fa \le E$. Now $a \vee E \ge a \vee fa \vee ef \ge a \vee f \vee e$. Therefore c is distinct from a,f,e and $ec \vee fc \le E$. If $d = f$ then $C \vee E \ge (cd \vee dc \vee ed \vee ec) \vee ae \ge a$. Therefore $d \ne f$. If $fd \le E$ then $C \vee E \ge (cd \vee dc \vee fc \vee fd) \vee ef \vee ae \ge a$. Therefore $df \le E$. If $d = e$ then $C \vee E \ge (cd \vee dc \vee fc \vee fd) \vee df \vee ad \ge a$. Therefore $d \ne e$. If $de \le E$ then $C \vee E \ge (ef \vee fe \vee de \vee df) \vee cd \vee ac \ge a$. Therefore $ed \le E$. Hence $C \vee E \ge (cd \vee dc \vee ec \vee ed) \vee ae \ge a$. We conclude $b\{a, (n+1) \vee C, (n+1) \vee E\} \le 2$.

<u>Part 4</u>. Let $\{A,C,E\} \subseteq \{A_f : f \in \quad\}$. Striving for a contradiction, assume $b\{(n+1) \vee A, (n+1) \vee C, (n+1) \vee E\} = 3$. Thus there must be $ab \le A$, $cd \le C$ and $ef \le E$ with $ab \nleq C \vee E$, $cd \nleq A \vee E$ and $ef \nleq A \vee C$. This implies that $ba \le C \wedge E$, $dc \le A \wedge E$ and $fe \le A \wedge C$.

<u>Statement 1</u>. $\{a,b\}$, $\{c,d\}$ and $\{e,f\}$ are disjoint doubletons.

<u>Proof</u>: By symmetry there are three cases to consider.

Case (i) - Assume $a = c$. $b \ne d$ since $ad \nleq A$ while $ab \le A$. If $bd \le C$ then $A \vee C \ge (ad \vee da \vee ba \vee bd) \vee ab \ge b \vee a \vee d$. If $db \le C$ then $A \vee C \ge (ab \vee ba \vee da \vee db) \vee ad \ge d \vee a \vee b$. Consequently a,b,d,e are distinct and $ae \vee be \vee de \le A \wedge C$. If $a = f$ then $A \vee E \ge (ef \vee fe \vee be \vee bf) \vee ab \ge a$. Therefore $a \ne f$. If $af \le C \vee E$ then $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. Therefore $fa \le C \wedge E$. If $b = f$ then $A \vee E \ge ef \vee fe \vee ae \vee af \ge a$.

Therefore $b \neq f$. If $fb \le A \vee E$ then $A \vee E \ge (ab \vee ba \vee fa \vee fb) \vee ef \vee ae \ge a$. Therefore $bf \le A \wedge E$. Hence $A \vee E \ge (ef \vee fe \vee be \vee bf) \vee ab \ge a$. We conclude $a \neq c$.

Case (ii) - Assume $a = d$. $b \neq c$ since $ab \nleq E$ while $ac \le E$. If $bc \le C$ then $A \vee C \ge (ac \vee ca \vee ba \vee bc) \vee ab \ge b \vee a \vee c$. If $cb \le C$ then $A \vee C \ge (ab \vee ba \vee ca \vee cb) \vee ac \ge c \vee a \vee b$. Consequently a,b,c,e are distinct and $ae \vee be \vee ce \le A \wedge C$. If $c = f$ then $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. Therefore $c \neq f$. If $cf \le C$ then $C \vee E \ge (ef \vee fe \vee ce \vee cf) \vee ac \ge a$. Therefore $fc \le C$. If $a = f$ then $C \vee E \ge (ae \vee ea \vee ca \vee ce) \vee ac \ge a$. Therefore $a \neq f$. If $fa \le C$ then $C \vee E \ge (ac \vee ca \vee fa \vee fc) \vee ef \vee ae \ge a$. Therefore $af \le C$. Hence $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. We conclude $a \neq d$.

Case (iii) - Assume $b = d$. $a \neq c$ since $cb \nleq A$ while $ab \le A$. If $ac \le C$ then $A \vee C \ge (bc \vee cb \vee ab \vee ac) \vee ba \ge a \vee b \vee c$. If $ca \le C$ then $A \vee C \ge (ab \vee ba \vee ca \vee cb) \vee bc \ge c \vee b \vee a$. Consequently a,b,c,e are distinct and $ae \vee be \vee ce \le A \wedge C$. If $a = f$ then $A \vee E \ge (ae \vee ea \vee ba \vee be) \vee ab \vee ce \ge c$. Therefore $a \neq f$. If $af \le E$ then $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. Therefore $fa \le E$. If $b = f$ then $A \vee E \ge (be \vee eb \vee ab \vee ae) \vee ba \vee ce \ge c$. Therefore $b \neq f$. If $fb \le E$ then $A \vee E \ge (ab \vee ba \vee fa \vee fb) \vee ef \vee ce \ge c$. Therefore $bf \le E$. Hence $A \vee E \ge (ef \vee fe \vee be \vee bf) \vee ab \vee fa \vee ce \ge c$. We conclude $b \neq d$. □

Statement 2. $x \in \{a,b,c,d\}$ implies $x \nleq A \vee C$.

Proof: Assume Statement 2 is false i.e. there exists $x \in \{a,b,c,d\}$ with $x \le A \vee C$.

Case (i) - $x \in \{a,b\}$. Then $A \vee C \ge ab \vee ba \vee x \ge a \vee b$. Therefore

$ae \vee be \le A \wedge C$. If $af \le E$ then $C \vee E \ge ef \vee fe \vee ae \vee af \ge a$. Therefore $fa \le E$. If $fb \le E$ then $A \vee E \ge (ab \vee ba \vee fa \vee fb) \vee ef \vee ae \ge f \vee e \vee a$. If $bf \le E$ then $A \vee E \ge (ef \vee fe \vee be \vee bf) \vee ab \vee fa \ge b \vee a \vee f \vee e$. Consequently $A \vee E \ge a \vee e \vee f$. Therefore $ac \vee ec \vee fc \le A \wedge E$. If $ad \le C$ then $C \vee E \ge cd \vee dc \vee ac \vee ad \ge a$. Therefore $da \le C$. Hence $A \vee C \ge a \vee da \vee cd \vee ec \ge e$. We conclude $x \nleq A \vee C$.

Case (ii) - $x \in \{c,d\}$. Then $A \vee C \ge cd \vee dc \vee x \ge c \vee d$. Therefore $ca \vee cb \le A \wedge C$. Otherwise the problem reduces to Case (i). It follows that $A \vee E \ge ab \vee ba \vee ca \vee cb \ge c$. We conclude $x \nleq A \vee C$. □

Statement 1 tells us that we may use condition (3) of our hypotheses on any i,j,k since they are in fact distinct. Statement 2 tells us that once there exists an $x \in \{a,b,c,d\}$ with $x \le A \vee C$ a contradiction has been reached.

Assume $db \le A$. If $da \le A \vee C$ then $A \vee C \ge ab \vee ba \vee da \vee db \ge d$. Therefore $ad \le A \wedge C$. If $ac \le A \vee C$ then $A \vee C \ge cd \vee dc \vee ac \vee ad \ge a$. Therefore $ca \le A \wedge C$. If $cb \le A \vee C$ then $A \vee C \ge ab \vee ba \vee ca \vee cb \ge c$. Therefore $bc \le A \wedge C$. If $bd \le C$ then $A \vee C \ge cd \vee dc \vee bc \vee bd \ge b$. Therefore $db \le C$. Hence $A \ge bc \vee ca \vee ad \vee db \vee ab \vee dc \vee fe$ and $C \ge bc \vee ca \vee ad \vee db \vee ba \vee cd \vee fe$. Analagously, if $bd \le A$ then $A \ge bd \vee da \vee ac \vee cb \vee ab \vee dc \vee fe$ and $C \ge bd \vee da \vee ac \vee cb \vee ba \vee cd \vee fe$. Note that up to now we have not used the order conditions (4) of the hypotheses. This allows us the freedom to identify these two cases under the permutations $\begin{pmatrix} a & b & c & d & e & f \\ c & d & a & b & e & f \end{pmatrix}$ and $\begin{pmatrix} A & C & E \\ C & A & E \end{pmatrix}$. Consequently we focus on the following situation:

$$A \ge bc \vee ca \vee ad \vee db \vee ab \vee dc \vee fe$$
$$C \ge bc \vee ca \vee ad \vee db \vee ba \vee cd \vee fe.$$

It is at this point that we use the order conditions (4).

(4) a) $i < j < k$ implies $i \leq ij \vee jk$

(4) b) $i < j < k < \ell$ implies $k \leq ik \vee ki \vee kj \vee j\ell$.

Case (i) – $a < b < c$. Then $A \geq ab \vee bc \geq a$. Contradiction.

Case (ii) – $a < c < b$. If $d < a$ then $A \vee C \geq dc \vee cd \vee ca \vee ab \geq c$. Contradiction.

If $a < d < b$ then $A \geq ad \vee db \geq a$. Contradiction. If $b < d$ then $A \vee C \geq ab \vee ba \vee bc \vee cd \geq b$. Contradiction.

Case (iii) – $b < a < c$. If $d < b$ then $A \geq db \vee bc \geq d$. Contradiction. If $b < d < a$ then $A \vee C \geq ba \vee ab \vee ad \vee dc \geq a$. Contradiction. If $a < d$ then $C \geq ba \vee ad \geq b$. Contradiction.

Case (iv) – $b < c < a$. Then $A \geq bc \vee ca \geq b$. Contradiction.

Case (v) – $c < a < b$. Then $A \geq ca \vee ab \geq c$. Contradiction.

Case (vi) – $c < b < a$. If $d < b$ then $C \geq db \vee ba \geq d$. Contradiction. If $b < d < a$ then $A \vee C \geq cd \vee dc \vee db \vee ba \geq d$. Contradiction. If $a < d$ then $A \geq ca \vee ad \geq c$. Contradiction.

Having reached contradictions through all possibilities, we conclude $b\{(n+1) \vee A, (n+1) \vee C, (n+1) \vee E\} \leq 2$. This completes the proof. □

CHAPTER III

Supercompactness and the cardinal function α

III.1. Introduction. A family of sets is *centered* if every finite subcollection has non-empty intersection. A family of sets is *linked* if the intersection of every pair of its members is non-empty. Alexander's lemma states that a space X is compact precisely when X possesses a closed subbase such that every centered subcollection has non-empty intersection. Paralleling this lemma, J. de Groot introduced the following definition in [11]. A space X is *supercompact* if X possesses a closed subbase such that every linked subcollection has non-empty intersection. Such a subbase is called a *binary* subbase. By Alexander's lemma, every supercompact space is compact.

Examples of supercompact spaces are plentiful. For a good introduction to supercompactness see A. Verbeek's book on superextensions [35]. It is shown there that any Tychonov space can be naturally embedded in many supercompact extensions. Every compact ordered space is supercompact by its left and right rays. A space is *treelike* if it is connected and every two points can be separated by a third. Brouwer and Schrijver [5] and J. van Mill [20] have shown that all compact treelike spaces are supercompact. De Groot proved that all compact polyhedra are supercompact. He conjectured that all compact metric spaces are supercompact. Several mathematicians - J. O'Connor [26], Strok and Szymánski [33], J. Martin and I. Rosenholtz [17] and E. van Douwen (in preparation) - have worked on this conjecture.

Products of supercompact spaces are again supercompact. Thus all Tychonov cubes I^{κ} and all Cantor cubes 2^{κ} are supercompact. Supercompactness has been instrumental in topologically characterizing Tychonov cubes and products of spheres, products of compact ordered spaces and products of compact treelike spaces. See Szymánski and Turzánski [34], de Groot and Schnare [12] and J. van Mill [20] respectively. In another vein, J. van Mill [21] has shown that the superextension of I is the Hilbert Cube, thus answering another conjecture of de Groot.

In [11] de Groot raises the question "Are all compact Hausdorff spaces supercompact?" At the time, A. Verbeek had an example of a compact T_1 non-supercompact space. We answer this question in the negative. Not only are counterexamples produced but positive implications are proven. Let βX denote the Stone-Cêch compactification of X. The following is shown: βX supercompact implies X is pseudocompact. Hence, neither βN, βQ nor βR are supercompact. This theorem is a particular consequence of a more general result which encompasses the case when two is replaced by any larger integer. Later in this chapter we investigate the cellular properties of a supercompact Hausdorff space and supply examples of first countable compact non-supercompact spaces. For supercompact spaces, the cellular results are much stronger than the previously mentioned theorem, however they do not generalize beyond two. Finally, a brief look at the Vietoris topology ends the chapter.

It is convenient to introduce the following cardinal function.

III.1.1. <u>Definition</u>. Let X be a topological space. $\alpha(X)$ is the least cardinal κ for which there exists an open subbase $\mathcal{S}$ of X with every cover of X from $\mathcal{S}$ having a subcover of size $< \kappa$. Equivalently,

$\alpha(X)$ is the least cardinal κ for which there exists a closed subbase $\mathcal{S}$ of X such that each subcollection $\mathcal{S}'$ of $\mathcal{S}$ with $\cap\mathcal{S}' = \emptyset$ has a subcollection $\mathcal{S}''$ with $\cap\mathcal{S}'' = \emptyset$ and $|\mathcal{S}''| < \kappa$. In either case we say that $\mathcal{S}$ *realizes* $\alpha(X) = \kappa$.

Clearly $\alpha(X) \leq 3$ iff X is supercompact and $\alpha(X) \leq \omega$ iff X is compact. It is tempting at this point to relate the Lindelöf property to $\alpha(X)$. The author thanks John Ginsburg for his comments on this matter.

III.1.2. <u>Proposition</u>. Let $\kappa > 1$. If for each $k \in K$, $\alpha(X_k) \leq \kappa$, then $\alpha\left(\prod_{k \in K} X_k\right) \leq \kappa$.

<u>Proof</u>: The proof is given for $X \times Y$. The general case is identical only messier. Suppose X has a subbase $\mathcal{S}$ and Y has a subbase $\mathcal{T}$ such that covers by members of $\mathcal{S}$ or $\mathcal{T}$ have subcovers of size $< \kappa$. Let $\mathcal{U} = \{S \times Y \colon S \in \mathcal{S}\} \cup \{X \times T \colon T \in \mathcal{T}\}$. Then $\mathcal{U}$ is a subbase for $X \times Y$. We claim covers by $\mathcal{U}$ have $< \kappa$ subcovers. For, let $\{S_i \times Y \colon i \in I\} \cup \{X \times T_j \colon j \in J\}$ cover $X \times Y$. Then either $X \subseteq \cup\{S_i \colon i \in I\}$ or $Y \subseteq \cup\{T_j \colon j \in J\}$. For if not, there exist $x_o \in X - \cup\{S_i \colon i \in I\}$ and $y_o \in Y - \cup\{T_j \colon j \in J\}$. But then (x_o, y_o) is not included in any of the sets $S_i \times Y$ or $X \times T_j$. Suppose w.l.o.g. that $X \subseteq \cup\{S_i \colon i \in I\}$. Find $I_1 \subseteq I$ such that $|I_1| < \kappa$ with $X \subseteq \cup\{S_i \colon i \in I_1\}$. Then $\{S_i \times Y \colon i \in I_1\}$ is a subcover of size $< \kappa$ as desired. $\square$

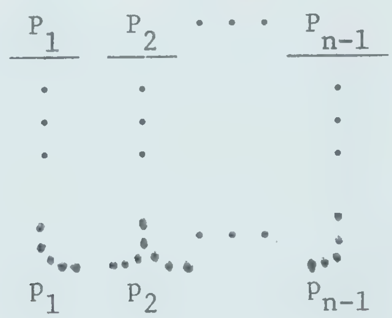
Now, one sees that $[\alpha(X) \leq \omega_1$ iff X is Lindelöf$]$ is false. Let Y be the Sorgenfrey line [32] and $X = Y \times Y$. Y is Lindelöf, hence $\alpha(Y) \leq \omega_1$ and thus $\alpha(X) \leq \omega_1$ by the proposition. However as is well-known, X is not Lindelöf.

It will follow from Theorem III.2.2 that α is not monotone even for a closed subspace of a compact Hausdorff space. More explicitly, $\alpha(I^c) = 3$, βN is a closed subspace of I^c but $\alpha(\beta N) = \omega$.

It is particularly interesting and in the spirit of de Groot to look at spaces X for which $\alpha(X)$ is finite.

III.1.3. <u>Example</u>. For each $3 < n < \omega$, we construct a compact T_1 space $X(n)$ with $\alpha\big(X(n)\big) = n$. $X(4)$ was originally constructed by A. Verbeek [35] as an example of a compact T_1 space which was not supercompact. The construction is an obvious extension of his idea.

Let $p_1, p_2, \cdots, p_{n-1}$ be $n-1$ distinct points. Let $P_i = \omega \times \{p_i\}$. Our underlying set is $X(n) = \bigcup_{i=1}^{n-1} P_i \cup \{p_1, p_2, \cdots, p_{n-1}\}$. For our topology, points of $\bigcup_{i=1}^{n-1} P_i$ are isolated and U is a neighbourhood of p_i iff $\bigcup_{j \neq i} (P_j - U)$ is finite. Thus, as a sequence, P_i converges to each p_j, where j is different than i.



A) $\alpha\big(X(n)\big) \leq n$.

First we note that $X(n)$, being the union of finitely many convergent sequences, is compact. We define open sets as follows: For $1 \leq i \leq n-1$ and $k \geq 0$, let $S_{ik} = \{p_i\} \cup \{(\ell, p_j) : \ell \geq k,\ j \neq i\}$ and $T_{ik} = \{p_i\} \cup \bigcup_{j \neq i} P_j \cup \{(\ell, p_i) : \ell \leq k\}$. Let $\mathcal{S} = \{S_{ik}, T_{ik} : 1 \leq i \leq n-1,\ k \geq 0\}$. Then $\mathcal{S}$ is our required subbase. That $\mathcal{S}$ is a subbase follows

from

1. $\{S_{ik}: k \geq 0\}$ is a local base at $\{p_i\}$.
2. For $1 \leq i \leq n-1$, $k \geq 0$, $\{(k,p_i)\} = \bigcap_{j \neq i} S_{ik} \cap \bigcap_{j} T_{jk}$.

From an arbitrary cover of $X(n)$ by members of $\mathcal{S}$, we first extract a finite subcover and from this an irreducible subcover. If the reader makes the following observations

1. For each $1 \leq i \leq n-1$, $\{S_{ik}: k \geq 0\}$ and $\{T_{ik}: k \geq 0\}$ are both chains under inclusion.
2. For each $1 \leq i \leq n-1$ and for all k and k', $S_{ik} \subseteq T_{ik'}$.
3. If $i \neq j$, then $p_i \notin S_{jk} \cup T_{jk'}$.

She or he will conclude that our irreducible subcover has the form $X(n) = \bigcup_{i \in A} S_{ik_i} \cup \bigcup_{j \in B} T_{j\ell_j}$ where $A \cup B = \{1,2,\cdots,n-1\}$ and $A \cap B = \emptyset$. Hence, the size of this subcover is $n-1$. What we have shown is that every irreducible cover of X from $\mathcal{S}$ has exactly $n-1$ elements.

B) $\alpha(X(n)) = n$.

It suffices to show that for an arbitrary subbase $\mathcal{S}$ of $X(n)$, there is an irreducible cover of $X(n)$ from $\mathcal{S}$ of size $\geq n-1$. For each $1 \leq i \leq n-1$ and $x \notin P_i \cup \{p_j: j \neq i\}$, there exists finitely many members of $\mathcal{S}$, each containing x, whose intersection misses $P_i \cup \{p_j: j \neq i\}$. Hence, one of these, $S(P_i,x)$ has the property that it misses infinitely many members of P_i, i.e. $\{p_j: j \neq i\} \cap S(P_i,x) = \emptyset$, while $x \in S(P_i,x)$. Consider $S(P_i,p_i)$. Since $p_i \in S(P_i,p_i)$, there are only finitely many points in $\bigcup_{j \neq i} P_j$ that are not in $S(P_i,p_i)$. Denote them by $x_1^i,\cdots,x_{n_i}^i$. Then $\bigcup_{j \neq i} P_j \subseteq S(P_i,p_i) \cup S(P_i,x_1^i) \cup \cdots \cup S(P_i,x_{n_i}^i)$. Thus $X(n) = \bigcup_{i=1}^{n-1} S(P_i,p_i) \cup \bigcup_{i=1}^{n-1} \left(\bigcup_{j=1}^{n_i} S(P_i,x_j^i)\right)$. Since p_i can only be

in a $S(P_i,-)$, $n-1$ are needed to cover $X(n)$. □

The question of whether there exists a compact Hausdorff space X with $\alpha(X) = n$ for $3 < n < \omega$ is postponed till later in the chapter.

III.2. Stone-Cêch Compactifications. Let X be a Tychonov space, βX the Stone-Cêch compactification of X and $Z(X)$ the collection of zero-sets of X. Two subsets of X said to be *completely separated* if they are contained in disjoint zero-sets.

Remark. At this point, let us note that if S is a closed subbase for X realizing $\alpha(X) = \kappa$ then w.l.o.g. we may assume that S is closed under finite intersections. Also, a collection S, closed under finite intersections, of closed subsets of a compact space is a closed subbase iff for each closed set C contained in an open set U there exists a finite subcollection F of S such that $C \subseteq \cup F \subseteq U$.

III.2.1. Lemma. Let S be a closed subbase for βX which is closed under finite intersections. Let U and V be completely separated subsets of X. Then there exist a finite $S' \subseteq S$ and a $Z \in Z(X)$ with $U \subseteq \cup S' \subseteq Cl_{\beta X}Z$ and $Z \cap V = \emptyset$.

Proof: Let $\{Z_1,Z_2\} \subseteq Z(X)$ such that $U \subseteq Z_1$, $V \subseteq Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Let $\{Z,Z'\} \subseteq Z(X)$ such that $Z \cap Z_2 = \emptyset$, $Z' \cap Z_1 = \emptyset$ and $Z \cup Z' = X$. Hence $Cl_{\beta X}Z' \cap Cl_{\beta X}Z_1 = \emptyset$ and $Cl_{\beta X}Z \cup Cl_{\beta X}Z' = \beta X$. Get a finite $S' \subseteq S$ such that $Cl_{\beta X}Z_1 \subseteq \cup S' \subseteq \beta X - Cl_{\beta X}Z' \subseteq Cl_{\beta X}Z$. S' and Z are as required. □

III.2.2. Theorem. If $\alpha(\beta X)$ is finite, then X is pseudocompact. In particular, for X non-pseudocompact βX is non-supercompact.

Proof: Assume $\alpha(\beta X) = N$ with $N > 2$. Let $\mathcal{S}$ be a closed subbase for βX, closed under finite intersections, realizing $\alpha(\beta X) = N$. Striving for a contradiction, assume X is not pseudocompact. Let $C = \{c_n : n < \omega\}$ be a subset of X such that there exists a continuous map f from X to R with $f(c_n) = n$. Define $C_n = \{x : n - \frac{1}{2} < f(x) < n + \frac{1}{2}\}$. Then $\mathcal{C} = \{C_n : n < \omega\}$ is a disjoint collection of cozero-sets of X with $c_n \in C_n$ and such that for each $A \subseteq \omega$, $\{c_n : n \in A\}$ and $X - \cup\{C_n : n \in A\}$ are completely separated.

For each $A \subseteq \omega$ apply Lemma III.2.1 with $U = \{c_n : n \in A\}$ and $V = X - \cup\{C_n : n \in A\}$ to yield a finite $\mathcal{S}_A \subseteq \mathcal{S}$ and a $Z_A \in \mathcal{Z}(X)$ with $\{c_n : n \in A\} \subseteq \cup\mathcal{S}_A \subseteq \mathrm{C}\ell\, Z_A$ and $Z_A \subseteq \cup\{C_n : n \in A\}$. Define $\mathcal{G} = \{f(S \cap C) : S \in \mathcal{S}_A \text{ and } A \subseteq \omega\}$ and $g : \mathcal{P}(\omega) \to [\mathcal{G}]^{<\omega}$ by $g(A) = \{f(S \cap C) : S \in \mathcal{S}_A\}$. Then $A = \cup g(A)$. Theorem I.11 implies there exist $\mathcal{H} \in [\mathcal{P}(\omega)]^N$ and for each $H \in \mathcal{H}$ a $G(H) \in g(H)$ such that

(1) $\cap\mathcal{H} = \emptyset$

(2) $\mathcal{H}' \in [\mathcal{H}]^{N-1}$ implies $\cap\{G(H) : H \in \mathcal{H}'\} \neq \emptyset$.

For each $H \in \mathcal{H}$ choose $S_H \in \mathcal{S}_H$ such that $G(H) = f(S_H \cap C)$.

The Contradiction. $\{S_H : H \in \mathcal{H}\}$ contradicts $\mathcal{S}$ realizing $\alpha(\beta X) = N$.

(a)
$$\begin{aligned}\cap\{S_H : H \in \mathcal{H}\} &\subseteq \cap\{\mathrm{C}\ell\, Z_H : H \in \mathcal{H}\}\\ &\subseteq \mathrm{C}\ell\left(\cap\{Z_H : H \in \mathcal{H}\}\right)\\ &\subseteq \mathrm{C}\ell\left(\cap\{\cup\{C_n : n \in H\} : H \in \mathcal{H}\}\right)\\ &= \emptyset.\end{aligned}$$

(b) Let $\mathcal{H}' \in [\mathcal{H}]^{N-1}$ and $n \in \cap\{G(H) : H \in \mathcal{H}'\} = \cap\{f(S_H \cap C) : H \in \mathcal{H}'\}$. Then $c_n \in \cap\{S_H : H \in \mathcal{H}'\}$.

Arriving at this contradiction we conclude that X is pseudocompact. □

If X is the Tychonov Plank, i.e. $X = ([0,\omega_1] \times [0,\omega]) - \{(\omega_1,\omega)\}$ then $\beta X = [0,\omega_1] \times [0,\omega]$ which is supercompact, while X is pseudocompact but not countably compact. Hence pseudocompact cannot be strengthened to countably compact.

The theorem determines the exact value of α for the following spaces: $\alpha(\beta N) = \alpha(\beta Q) = \alpha(\beta R) = \omega$.

βN is a non-supercompact subspace of I^c which is supercompact. Hence supercompactness is not a closed hereditary property and α is not monotone.

An *extremally disconnected* (E.D.) space has the property that disjoint open sets have disjoint closures. These arise quite naturally since in the category of compact spaces and continuous maps these are precisely the projective elements. A compact Hausdorff E.D. space is the Stone-Cêch compactification of each of its dense subspaces. For a proof of this, the reader is referred to Gillman and Jerison [10] pg. 96. Since no infinite Hausdorff space can have all of its dense subspaces pseudocompact it follows that an infinite compact Hausdorff E.D. space X satisfies $\alpha(X) = \omega$.

All compact non-supercompact spaces derived from the above theorem have cardinality at least 2^c. They all contain a copy of βN. The next section produces smaller examples.

III.3. <u>A Cellular Constraint in Supercompact Spaces</u>. Further work on supercompact spaces X $(\alpha(X) \leq 3)$ has since been done by Eric van Douwen and Jan van Mill [6]. The author expresses thanks to these two gentlemen for sending copies of their work. They have proven the following two theorems:

[A] (J. van Mill). Let X be a supercompact Hausdorff space. If Y is a continuous image of a closed neighbourhood retract of

X then for all countably infinite subsets K of Y all but countably many cluster points of K are the limit of some non-trivial sequence in Y (not necessarily in K).

[B] (E. van Douwen). Let X be a supercompact Hausdorff space. Then no closed neighbourhood retract of X is homeomorphic to any compactification of a κ-Cantor tree (cf. M.E. Rudin [29]) where $\omega < \kappa \leq c$.

We remark that J. van Mill used [A] to give a different proof that βX supercompact implies X pseudocompact, furthermore he showed that no infinite compact F-space (disjoint cozero-sets are completely separated) is supercompact. E. van Douwen used [B] to construct a compact Hausdorff non-supercompact space of cardinality ω_1 and a compact Hausdorff non-supercompact first countable space of cardinality c. However, there are compact Hausdorff non-supercompact spaces which are not covered by these results and it is to this end that this section is devoted.

III.3.1. <u>Lemma</u>. Let X be a subspace of weight κ of a space Y. Let $\{V_\alpha: \alpha < \kappa^+\}$ and $\{U_\alpha: \alpha < \kappa^+\}$ be open sets of Y such that

(1) For $\alpha < \beta < \kappa^+$, $C\ell_Y V_\alpha \cap C\ell_Y V_\beta$ is a compact set of X.

(2) For $\alpha < \kappa^+$, $C\ell_Y V_\alpha \subseteq U_\alpha$.

Then there exists $\{U_n: n \geq 0\} \subseteq \{U_\alpha: \alpha < \kappa^+\}$ (relabelled for convenience) such that $\sup\{m: U_o$ contains all 2 fold intersections of m $C\ell_Y V_n$'s, $n \geq 1\} = \omega$.

<u>Proof</u>: Let $\mathcal{B}$ be an open base for X, closed under finite unions, of cardinality κ. For $\alpha < \beta < \kappa^+$ choose $B_{\alpha\beta} \in \mathcal{B}$ such that $C\ell_Y V_\alpha \cap C\ell_Y V_\beta \subseteq B_{\alpha\beta} \subseteq U_\alpha \cap U_\beta$. For $2 \leq m < \omega$ define $D_m = \{\alpha < \kappa^+: U_\alpha$ contains all 2 fold intersections of m other $C\ell_Y V_\beta$'s$\}$. It suffices

to show that for each $m \geq 2$, $|\{\alpha < \kappa^+ : \alpha \notin D_m\}| \leq \kappa$. For then, just choose $0 \in \cap\{D_m : m \geq 2\}$ and the corresponding finite collections to make up the U_n's for $n \geq 1$. To this end the proof for D_3 is given, the general case is identical only longer.

Assume $F_o = \{\alpha < \kappa^+ : \alpha \notin D_3\}$ has cardinality κ^+. Choose $\alpha \in F_o$ and consider $\{B_{\alpha\beta} : \beta \in F_o - \{\alpha\}\}$. There exists $F_1 \subseteq F_o - \{\alpha\}$, $|F_1| = \kappa^+$, such that for distinct β and γ in F_1, $B_{\alpha\beta} = B_{\alpha\gamma}$. Choose $\beta \in F_1$ and consider $\{B_{\beta\gamma} : \gamma \in F_1 - \{\beta\}\}$. There exists $F_2 \subseteq F_1 - \{\beta\}$, $|F_2| = \kappa^+$, such that for distinct γ and δ in F_2, $B_{\beta\gamma} = B_{\beta\delta}$. Choose distinct γ and δ from F_2. Then $(C\ell_Y V_\alpha \cap C\ell_Y V_\beta) \cup (C\ell_Y V_\alpha \cap C\ell_Y V_\gamma) \cup (C\ell_Y V_\beta \cap C\ell_Y V_\gamma) \subseteq B_{\alpha\beta} \cap B_{\alpha\gamma} \cap B_{\beta\gamma} \subseteq U_\delta$. Hence $\delta \in D_3$, a contradiction. Consequently $|F_o| \leq \kappa$. □

III.3.2. <u>Theorem</u>. Let X be a supercompact Hausdorff space. If K is a closed neighbourhood retract of X, then for all dense D in K, $c(K-D) \leq w(D)$.

<u>Proof</u>: Let S be a binary closed subbase for X that is closed under finite intersections. Let r be a retraction of an open set U onto K. Assume $w(D) = \kappa$ and choose a dense subset E of D with $|E| \leq \kappa$. Striving for a contradiction, assume $c(K-D) > \kappa$. Let $\{C_\alpha : \alpha < \kappa^+\}$ be an open cellular family in $K - D$. Pick $p_\alpha \in C_\alpha$ and since K is regular choose an open set W_α of K such that $p_\alpha \in W_\alpha$ and $C\ell_K W_\alpha \cap (K-D) \subseteq C_\alpha$. Using the normality of K, find $E_\alpha \subseteq E$ and open sets V_α and U_α of K such that $p_\alpha \in C\ell_K E_\alpha \subseteq V_\alpha \subseteq C\ell_K V_\alpha \subseteq U_\alpha \subseteq C\ell_K U_\alpha \subseteq W_\alpha$. Notice that $C\ell_K E_\alpha$ is a closed set of X contained in the open set $r^{-1}(V_\alpha)$ of X. Using the fact that S is a subbase, get $S_\alpha \in S$ such that $S_\alpha \subseteq r^{-1}(V_\alpha)$ and $p_\alpha \in C\ell_K(E_\alpha \cap S_\alpha)$. Let $F_\alpha = E_\alpha \cap S_\alpha$. Since $|E| \leq \kappa$ and there are κ^+

α's w.l.o.g. assume there exists $f \in \cap\{F_\alpha : \alpha < \kappa^+\}$.

For $\alpha < \beta < \kappa^+$, $C\ell_K V_\alpha \cap C\ell_K V_\beta$ is a compact subset of D, for if not, then it would meet $K - D$ which is impossible since $C_\alpha \cap C_\beta = \emptyset$. It follows from Lemma III.3.1 that there exists $\{U_n : n \geq 0\} \subseteq \{U_\alpha : \alpha < \kappa^+\}$ such that $\sup\{m: U_o$ contains all 2 fold intersections of m $C\ell_K V_n$'s, $n \geq 1\} = \omega$. Because $C\ell_K W_o \cap \cup\{C\ell_K U_n \cap (K-D) : n \geq 1\} = \emptyset$, there exists U open in K such that $U \cap C\ell_K W_o = \emptyset$ and $\cup\{C\ell_K U_n \cap (K-D) : n \geq 1\} \subseteq U$. For $n \geq 1$, pick $f_n \in (U \cap F_n) - U_o$. This is possible since U is an open neighbourhood of p_n, $p_n \in C\ell_K F_n$ and $U \cap F_n \cap U_o$ is contained in the compact set $C\ell_K U_n \cap C\ell_K U_o$ of D. Notice that

$$C\ell_K\{f_n : n \geq 1\} \subseteq C\ell_K U \subseteq C\ell_K(K - C\ell_K W_o) \subseteq K - W_o \subseteq K - C\ell_K U_o .$$

Thus, $C\ell_K\{f_n : n \geq 1\}$ is a closed set of X contained in the open set $r^{-1}(K - C\ell_K U_o)$ of X. Hence, there exists $\{S_i : 1 \leq i \leq k\} \subseteq \mathcal{S}$ with $C\ell_K\{f_n : n \geq 1\} \subseteq \cup\{S_i : 1 \leq i \leq k\} \subseteq r^{-1}(K - C\ell_K U_o)$. Choose $k+1$ $C\ell_K V_n$'s such that U_o contains all 2 fold intersections of the $C\ell_K V_n$'s. By the pigeon hole principle, there exist $n \neq m$ and an $1 \leq i \leq k$ such that $\{f_n, f_m\} \subseteq S_i$ and $C\ell_K V_n \cap C\ell_K V_m \subseteq U_o$.

<u>The Contradiction</u>. $\{S_n, S_m, S_i\}$ is linked yet has empty intersection.

$$f \in F_n \cap F_m \subseteq S_n \cap S_m$$
$$f_n \in F_n \cap S_i \subseteq S_n \cap S_i$$
$$f_m \in F_m \cap S_i \subseteq S_m \cap S_i .$$

However,
$$\begin{aligned} S_n \cap S_m \cap S_i &\subseteq r^{-1}(V_n) \cap r^{-1}(V_m) \cap r^{-1}(K - C\ell_K U_o) \\ &\subseteq r^{-1}\big(V_n \cap V_m \cap (K - C\ell_K U_o)\big) \\ &= \emptyset . \ \square \end{aligned}$$

Notice that [B] is a consequence of Theorem III.3.2 since any

compactification X of a κ-Cantor tree $(\omega < \kappa \leq c)$ is a compactification of N with $c(X-N) > \omega$ while $w(N) = \omega$. Both [A] and this theorem show that for X non-pseudocompact, neither βX nor $\beta X - X$ (when X is locally compact) are supercompact since in this case both βX and $\beta X - X$ contain a neighbourhood retract homeomorphic to βN. To see this, let $\{Z'_n: n < \omega\}$, $\{C_n: n < \omega\}$ and $\{Z_n: n < \omega\}$ be three collections of non-compact zero-sets of X, cozero-sets of X and zero-sets of X respectively, such that

(1) for each $n < \omega$, $Z'_n \subseteq C_n \subseteq Z_n$

(2) for $n \neq m$, $Z_n \cap Z_m = \emptyset$

(3) for each $A \subseteq \omega$, $\bigcup_{n\in A} Z'_n$ is a zero-set, $\bigcup_{n\in A} C_n$ is a cozero-set and $\bigcup_{n\in A} Z_n$ is a zero-set.

Then $X = \bigcup_{n<\omega} Z_n \cup \left(X - \bigcup_{n<\omega} C_n\right)$. Choose $c_n \in C_n$ and $p_n \in C\ell_{\beta X} Z'_n - Z'_n$. $\{c_n: n < \omega\}$ ($\{p_n: n < \omega\}$) is C^*-embedded in X (βX), hence $C\ell_{\beta X}\{c_n: n < \omega\}$ ($C\ell_{\beta X}\{p_n: n < \omega\}$) is homeomorphic to βN. The map $r: \beta X - C\ell_{\beta X}\left(X - \bigcup_{n<\omega} C_n\right) \to C\ell_{\beta X}\{c_n: n < \omega\}$ defined by $r(p) = \cap\left\{C\ell_{\beta X}\{c_n: n \in A\}: \bigcup_{n\in A} Z_n \in p\right\}$ is a retraction of an open set of βX onto a copy of βN. The map $r': (\beta X - X) - C\ell_{\beta X}\left(X - \bigcup_{n<\omega} C_n\right) \to C\ell_{\beta X}\{p_n: n < \omega\}$ defined by $r'(p) = \cap\left\{C\ell_{\beta X}\{p_n: n \in A\}: \bigcup_{n\in A} Z_n \in p\right\}$ is a retraction of an open set of $\beta X - X$ onto a copy of βN.

III.3.3. <u>Example</u>. In the theorem, cellularity cannot be replaced by spread. J. van Mill [22] has shown (in particular) that there exists a supercompactification γN of N such that $\gamma N - N = 2^c$. Since 2^c con-

tains a copy of βN, $s(2^c) = c$. Hence, $s(\gamma N - N) \nleq w(N)$.

III.3.4. Example. In the theorem, weight cannot be replaced by density. Let βN be a copy of βN in 2^c. Let $D = N \cup (2^c - \beta N)$. Then D is open in 2^c and therefore separable i.e. $d(D) = \omega$. However, $c(2^c - D) = c(\beta N - N) = c$. Hence, $c(2^c - D) \nleq d(D)$.

Our concern now is to give examples of compact Hausdorff non-supercompact spaces not covered by [A] or [B]. They will all be first countable compactifications γN of N such that $\gamma N - N$ is connected and locally connected. We use the following result of E. van Douwen and T.C. Przymusínski [7].

[C] Let $\mathcal{B}$ be an open base for a compact Hausdorff space Y with $|\mathcal{B}| \leq c$, $Y \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Assume there is a function $h\colon \mathcal{B} \to \mathcal{P}(N)$ such that

(0) $h(Y) = N$

(1) $h(B)$ is infinite for $B \in \mathcal{B}$

(2) if $A, B \in \mathcal{B}$ are disjoint, then $h(A) \cap h(B)$ is finite

(3) if $A \in \mathcal{B}$ and if $\mathcal{F} \subseteq \mathcal{B}$ is finite, and if $A \subseteq \cup\mathcal{F}$, then $h(A) - \cup\{h(B)\colon B \in \mathcal{F}\}$ is finite.

Then there is a Hausdorff compactification γN of N such that $\gamma N - N$ and Y are homeomorphic. Furthermore γN is first countable if Y is first countable.

Indeed, as the authors show, the family $\{B \cup (h(B) - F)\colon B \in \mathcal{B},\ F \subseteq N \text{ finite}\} \cup \{\{n\}\colon n \in N\}$ is an open base for the required topology on $\gamma N = N \uplus Y$.

We introduce the following construction which generalizes the

Alexandrov duplicate of a space. Let $f\colon X \to Y$ where X and Y are T_1 spaces. Define $X f Y$ to be the space with underlying set $X \times Y$ and topology as follows: Basic open neighbourhoods of (x,y) where $y \neq f(x)$ are of the form $\{x\} \times (U_y - \{f(x)\})$ where U_y is an open neighbourhood of y in Y. Basic open neighbourhoods of $(x, f(x))$ are of the form $[(U_x - \{x\}) \times Y] \cup [\{x\} \times U_{f(x)}]$ where U_x $(U_{f(x)})$ is an open neighbourhood of $x(f(x))$ in $X(Y)$. $X f Y$ is a T_1 space. Properties that $X f Y$ inherits from both X and Y include compactness, Hausdorffness, connectedness, local connectedness and first countability. Furthermore, the following cardinal inequalities hold if $|Y| > 1$; $c(X f Y) = |X| \cdot c(Y)$ and $w(X f Y) = |X| \cdot w(X) \cdot w(Y)$.

Recall that a space X is *sequentially separable* if it has a countable dense subset D such that every point of X is the limit of some convergent sequence of points from D.

III.3.5. Proposition. Let X be an infinite sequentially separable compact Hausdorff space with no isolated points and Y be a separable compact Hausdorff space with no isolated points. Then there exists a Hausdorff compactification γN of N such that $\gamma N - N$ and $X f Y$ are homeomorphic (for any f).

Proof: Let $\mathcal{U}$ be a base for X with $|\mathcal{U}| \leq c$, $\emptyset \notin \mathcal{U}$ and $\mathcal{V}$ be a base for Y with $|\mathcal{V}| \leq c$, $\emptyset \notin \mathcal{V}$. Since X is sequentially separable, note that $|X| \leq c$. Define $\mathcal{B}_1 = \{\{x\} \times (V - \{f(x)\}) : x \in X, V \in \mathcal{V}\}$ and $\mathcal{B}_2 = \{[(U - \{x\}) \times Y] \cup [\{x\} \times V] : x \in U \in \mathcal{U}$ and $f(x) \in V \in \mathcal{V}\}$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{X f Y\}$. $\mathcal{B}$ is a base for the open sets of $X f Y$ with $|\mathcal{B}| \leq c$. Let D be a countable dense subset of X rendering X sequentially separable and E be a countable dense subset of Y. Let

$g: N \to D$ be a bijection. For each $x \in X$ choose a non-trivial sequence $\{d_k^x: k < \omega\} \subseteq D$ such that (d_k^x) converges to x. Consider $N_x = \{n: g(n) \in \{d_k^x: k < \omega\}\}$. $\{N_x: x \in X\}$ is an almost disjoint family. For each $x \in X$, let $g_x: N_x \to E$ be a bijection.

For $x \in X$ and $V \in \mathcal{V}$ define

$$h(\{x\} \times (V - \{f(x)\})) = \{n: n \in N_x \text{ and } g_x(n) \in E \cap V\}.$$

For $x \in U \in \mathcal{U}$ and $f(x) \in V \in \mathcal{V}$ define

$$\begin{aligned} h([(U - \{x\}) \times Y] \cup [\{x\} \times V]) = \{n: n \notin N_x &\text{ and } g(n) \in D \cap U\} \cup \\ \{n: n \in N_x &\text{ and } g_x(n) \in E \cap V\}. \end{aligned}$$

Define $h(X f Y) = N$. It is now a straightforward exercise that $h: \mathcal{B} \to \mathcal{P}(N)$ satisfies all the hypotheses of [C] and thus there exists a compactification γN of N such that $\gamma N - N$ and $X f Y$ are homeomorphic. □

Hence if X and Y are infinite Peano spaces (compact, connected, locally connected metrizable spaces) and $f: X \to Y$ is an arbitrary correspondence, then γN where $\gamma N - N = X f Y$ is an example of a compact Hausdorff first countable non-supercompact (since $c(X f Y) = c$) space.

We remark that two further examples can be found in the theory of lexicographic order. Consider the long line and the lexicographic ordered square (cf. S. Willard [37]). Both have cellularity $> \omega$ and both are remainders of N in some compactification. These compactifications aren't supercompact.

III.3.6. <u>Example</u>. A compact Hausdorff space X with $\alpha(X) = 4$. This example is the complete c-Cantor tree compactified by adding one point. Let ${}^{\omega}2 = \{f: f: \omega \to 2\}$ and ${}^{\underline{\omega}}2 = \{f: \text{there exists } n < \omega \text{ and } f: n \to 2\}$. $T = {}^{\omega}2 \cup {}^{\underline{\omega}}2$ is called the Cantor tree. For each $f \in {}^{\omega}2$,

let $I(f) = \{f|n: n < \omega\}$. The topology on T is as follows: Points of ${}^{\omega}2$ are isolated and neighbourhoods of $f \in {}^{\omega}2$ contain f and all but finitely many members of $I(f)$. T is first countable and locally compact. Let X be the one point compactification of T. Since $w({}^{\omega}2) = \omega$ and $c(X - {}^{\omega}2) = c$ it follows that X is not supercompact, i.e. $\alpha(X) > 3$.

For $f \in {}^{\omega}2$ and $n < \omega$, let $S(f,n) = \{f\} \cup \{f|k: n \leq k < \omega\}$. For a finite subset H of ${}^{\omega}2$, let $S(H) = X - \cup\{S(f,0): f \in H\}$. Let $\mathcal{S} = \{S(f,n): f \in {}^{\omega}2 \text{ and } n < \omega\} \cup \{S(H): H \text{ is a finite subset of } {}^{\omega}2\} \cup \{\{g\}: g \in {}^{\omega}2\}$. $\mathcal{S}$ is a closed subbase for X realizing $\alpha(X) = 4$. That $\mathcal{S}$ is a closed subbase is true because X is compact and 0-dimensional and each clopen set is a finite union of members of $\mathcal{S}$. Let $\mathcal{S}' \subseteq \mathcal{S}$ with $\cap\mathcal{S}' = \emptyset$. By compactness, there exists a finite $\mathcal{S}'' \subseteq \mathcal{S}'$ with $\cap\mathcal{S}'' = \emptyset$. If $\mathcal{S}''$ contains a singleton, then there will be two members of $\mathcal{S}''$ which are disjoint. So, assume $\mathcal{S}''$ does not contain a singleton. The following facts are easily verified:

<u>Fact 1</u>. $\{S(f,n): f \in {}^{\omega}2 \text{ and } n < \omega\}$ has the property that the intersection of three is actually the intersection of two of them.

<u>Fact 2</u>. $S(f_1,n_1) \cap S(f_2,n_2) \subseteq \bigcup_{i=1}^{n} S(g_i,0)$ implies there exists $1 \leq i \leq n$ such that $S(f_1,n_1) \cap S(f_2,n_2) \subseteq S(g_i,0)$.

<u>Fact 3</u>. $S(f,n) \subseteq \bigcup_{i=1}^{n} S(g_i,0)$ implies there exists $1 \leq i \leq n$ such that $S(f,n) \subseteq S(g_i,0)$. Actually, f will equal a g_i.

Facts 1 thru 3 lead to, an at most, three element subset of $\mathcal{S}''$ whose total intersection is empty. This shows $\alpha(X) \leq 4$. Combining the two inequalities, we conclude that $\alpha(X) = 4$.

This example also shows that $\alpha(X) \leq 3$ in Theorem III.3.2 cannot be strengthened to $\alpha(X) < \omega$. The author has not been able to generalize this example beyond 4.

III.4. Supercompactness in the Vietoris Topology. Our attention is now diverted to the exponential or finite topology of Vietoris [36]. The author would like to thank A. Arhangel'skii for his motivation in this direction. An excellent background on the Vietoris topology can be found in E. Michael's paper [19]. For a space X, let $\mathrm{Exp}(X)$ denote the collection of all non-empty closed subsets of X. Let $\langle U_1,\cdots,U_n\rangle = \{F \in \mathrm{Exp}(X)\colon F \subseteq \cup\{U_i\colon 1 \leq i \leq n\}$ and for each $1 \leq i \leq n$, $F \cap U_i \neq \emptyset\}$. Then $\{\langle U_1,\cdots,U_n\rangle\colon$ for $1 \leq i \leq n$, U_i is open in $X\}$ serves as a base for the open sets of $\mathrm{Exp}(X)$. For $A \subseteq X$, let $\mathrm{Exp}(A) = \{F \in \mathrm{Exp}(X)\colon F \subseteq A\}$. Then $\mathrm{Exp}(A)$ is open (closed) in $\mathrm{Exp}(X)$ if A is open (closed) in X. In [19], E. Michael has shown that $\mathrm{Exp}(X)$ is compact Hausdorff iff X is compact Hausdorff.

Let X be compact Hausdorff. If Y is a closed neighbourhood retract of X, then $\mathrm{Exp}(Y)$ is a closed neighbourhood retract of $\mathrm{Exp}(X)$. To see this, let $r\colon U \to Y$ be a retraction of some open set U of X onto Y. Then the map $r'\colon \mathrm{Exp}(U) \to \mathrm{Exp}(Y)$ defined by $r'(F) = \{r(x)\colon x \in F\}$ is a retraction. Note that F is a compact subset of X contained in U, hence $r'(F)$ is closed in X.

The following gives a necessary condition that $\mathrm{Exp}(X)$ be supercompact.

III.4.1. Proposition. Let $\mathrm{Exp}(X)$ be a supercompact Hausdorff space. Then for all D dense in a closed neighbourhood retract Y of X, $c(Y-D) \leq w(D)$.

Proof: Let V be an open set in X which retracts onto Y. Let $C(D) = \{$all compact subsets of $D\}$. Then $C(D)$ is dense in the closed retract $\mathrm{Exp}(Y)$ of $\mathrm{Exp}(V)$ (actually the finite subsets of D are

dense in $\mathrm{Exp}(Y)$, however this subspace is too small for our purposes). Hence, by Theorem III.3.2, $c\big(\mathrm{Exp}(Y) - \mathcal{C}(D)\big) \leq w\big(\mathcal{C}(D)\big)$. But $c(Y-D) \leq c\big(\mathrm{Exp}(Y) - \mathcal{C}(D)\big)$. To see this, note that if $\mathcal{U}$ is a collection of open sets of Y such that $\{U \cap (Y-D) : U \in \mathcal{U}\}$ is cellular in $Y-D$, then $\{\mathrm{Exp}(U) : U \in \mathcal{U}\}$ is a collection of open sets of $\mathrm{Exp}(Y)$ such that $\big\{\mathrm{Exp}(U) \cap \big(\mathrm{Exp}(Y) - \mathcal{C}(D)\big) : u \in \mathcal{U}\big\}$ is cellular in $\mathrm{Exp}(Y) - \mathcal{C}(D)$. Also, if $\mathcal{B}$ is an open base for D, closed under finite unions, of cardinality $w(D)$, then

$$\big\{<\mathrm{Int}_Y(C\ell_Y B_1),\cdots,\mathrm{Int}_Y(C\ell_Y B_n)> \cap\ \mathcal{C}(D) : \{B_1,\cdots,B_n\} \subseteq \mathcal{B}\big\}$$

is an open base for $\mathcal{C}(D)$ of cardinality $w(D)$. Hence $w\big(\mathcal{C}(D)\big) = w(D)$. Consequently, $c(Y-D) \leq w(D)$. □

It now follows that spaces like $\mathrm{Exp}(\beta N)$, $\mathrm{Exp}(\beta N-N)$ and $\mathrm{Exp}(\gamma N)$ (where $\gamma N - N$ is the long line) are non-supercompact.

III.4.2. Outstanding problems in Supercompactness. Considering

(a) The spaces 2^κ are the simplest supercompact spaces,

(b) All the T_2 continuous images of 2^ω are supercompact (i.e. all compact metric spaces),

(c) For X non-pseudocompact, βX is neither dyadic (a continuous image of some 2^κ) nor supercompact,

the conjecture that all dyadic compacta are supercompact is reasonable. Note that $[0,\omega_1]$ is supercompact but not dyadic, so these two concepts (both relating to the number 2) are distinct. The question whether supercompactness is transferred to continuous T_2 images is unsolved. Indeed, it is not even known if $X \times X$ supercompact implies X is supercompact. Also it is unknown whether supercompactness is passed from a space to a closed G_δ subspace or to a neighbourhood retract.

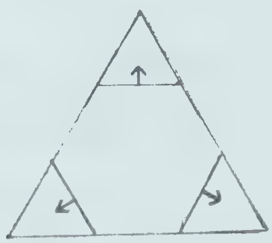
For each n such that $3 < n < \omega$, does there exist a first countable compact T_2 space X with $\alpha(X) = n$? Does $\mathrm{Exp}(X)$ supercompact imply X supercompact?

CHAPTER IV

Breadth in Topological Spaces

IV.1. Introduction. While the cardinal function $\alpha(X)$ defined in Chapter III is productive, i.e. $\alpha\left(\prod_{i\in I} X_i\right) \leq \sup\{\alpha(X_i): i \in I\}$, it is not monotone, i.e. $A \subseteq X$ does not imply $\alpha(A) \leq \alpha(X)$ necessarily. In this chapter we look at a related property which is monotone but not productive in the above sense.

Consider the usual open subbase for the closed unit square I^2, $\mathcal{S} = \{[0,\alpha) \times I: 0 < \alpha \leq 1\} \cup \{(\alpha,1] \times I: 0 \leq \alpha < 1\} \cup \{I \times [0,\alpha): 0 < \alpha \leq 1\} \cup \{I \times (\alpha,1]: 0 \leq \alpha < 1\}$. $\mathcal{S}$ renders I^2 supercompact, i.e. every open cover of I^2 from $\mathcal{S}$ has a two subcover. Looking again at $\mathcal{S}$, we see that it enjoys further properties. In particular, $\mathcal{S}$, being the union of four chains, satisfies the following: Every union of five members of $\mathcal{S}$ is actually a union of four of them. Can we do better for I^2? Sure we can. I^2 is homeomorphic to the two-simplex. Take the subbase consisting of the three different types of open subsets as suggested in the following diagram.



Thus we see that I^2 has a subbase $\mathcal{S}$ such that every union of four members of $\mathcal{S}$ is actually a union of three of them. It is an open question at the time of writing whether this is best possible.

<u>Open Question</u>. Does there exist an open subbase for I^2 such that the union of three is actually the union of two of them?

In general, the n-cube I^n has a subbase such that the union of $n+2$ of them is a union of $n+1$ of them. With this in mind, we make the following definition. Consider $(\mathcal{P}(X), \cup, \cap, ', \emptyset, X)$ as a boolean lattice. Recall from Chapter II that if $\mathcal{S} \subseteq \mathcal{P}(X)$ then the breadth of $\mathcal{S}$, $b(\mathcal{S})$, is the smallest positive integer b (if one exists) such that any union $S_1 \cup S_2 \cup \cdots \cup S_{b+1}$ (with $S_i \in \mathcal{S}$) is always a union of b of the S_i.

IV.2. <u>Definition</u>. Let X be a topological space. The *breadth of* X, $b(X)$, is the smallest positive integer b (if one exists) such that X has an open subbase $\mathcal{S}$ with $b(\mathcal{S}) = b$. If no such b exists, we say X has infinite breadth or $b(X) = \infty$.

As with the cardinal function $\alpha(X)$, we are particularly interested in spaces with breadth two. But first of all, let us establish several immediate consequences of the definition.

IV.3. <u>Proposition</u>. (a) $A \subseteq X$ implies $b(A) \leq b(X)$ (b is monotone)

(b) $b(X \times Y) \leq b(X) + b(Y)$.

<u>Proof</u>: (a) If $b(X)$ is ∞, then we are finished. So assume $b(X) = n$. Let $\mathcal{S}$ be a subbase for X with $b(\mathcal{S}) = n$. Then $\{A \cap S\colon S \in \mathcal{S}\}$ is a subbase for A of breadth $\leq n$.

(b) If either $b(X)$ or $b(Y)$ is ∞, then we are finished. So assume $b(X) = n$, $\mathcal{S}$ is a subbase for X with $b(\mathcal{S}) = n$, $b(Y) = m$ and $\mathcal{T}$ is a subbase for Y with $b(\mathcal{T}) = m$. Define $\mathcal{R} = \{S \times Y\colon S \in \mathcal{S}\} \cup \{X \times T\colon T \in \mathcal{T}\}$. Then $\mathcal{R}$ is a subbase for $X \times Y$ and $b(\mathcal{R}) = n+m$. Hence

$b(X \times Y) \leq n+m$.

IV.4. Examples.

(i) Identifying the n-cube with the n-simplex, we have $b(I^n) \leq n+1$.

(ii) All totally ordered spaces have breadth two.

(iii) Since all n-dimensional separable metric spaces can be imbedded in a $2n+1$-simplex (see Hurewicz and Wallman [14]), for such spaces X, $b(X) \leq 2n+2$.

(iv) The compact T_1 spaces, $X(n)$, of III.1.3 have breadth $n-1$.

IV.5. Relationship of $b(X)$ and $\alpha(X)$. If $\alpha(X) \leq \omega$, then $\alpha(X) \leq b(X)+1$. Certainly, if $b(X)$ is ∞, this is true. Otherwise, let $b(X) = n$ and S be an open subbase for X with $b(S) = n$. Then any cover of X from S has a finite subcover and since a cover is a union, this can be reduced to at most n members, whence $\alpha(X) \leq n+1$. An example where strict inequality holds is $X = 2^c$. By Theorem III.2.2, $\alpha(\beta N) = \omega$. So by the above inequality $b(\beta N) = \infty$, therefore by monotonicity of b, $b(2^c) = \infty$ (βN is a subspace of 2^c). But 2^c is supercompact hence $\alpha(2^c) = 3$. Consequently $\alpha(2^c) < b(2^c) + 1$. An example where equality obtains in $X = I$.

We now proceed towards our main objective which is to prove that all one-dimensional separable metric spaces have breadth two (we exclude the one point space which has breadth one). Clearly, all zero-dimensional separable metric spaces, being subspaces of the totally ordered Cantor discontinuum, have breadth two. This result will be a strengthening of the gross inequality of IV.4(iii) for $n = 1$.

Henceforth we make the blanket assumption:

X DENOTES A ONE-DIMENSIONAL SEPARABLE METRIC SPACE.

If $\mathcal{B} \subseteq \mathcal{P}(X)$ then by $[\mathcal{B}]$ we mean the ring generated by $\mathcal{B}$ i.e. we close $\mathcal{B}$ under finite unions and finite intersections. As usual, $\mathrm{Bd}_X B = \mathrm{C}\ell_X B \cap \mathrm{C}\ell_X(X-B)$ (the boundary of B). The subscript X will be suppressed when the meaning is clear. We mention two facts.

(a) B open in X implies $\mathrm{Bd}B = \mathrm{C}\ell B - B$

(b) $\mathcal{F}$ a finite subcollection of $\mathcal{B}$ implies $\mathrm{Bd}\left(\cap\{B\colon B \in \mathcal{F}\}\right) \subseteq \cup\{\mathrm{Bd}B\colon B \in \mathcal{F}\}$ and $\mathrm{Bd}\left(\cup\{B\colon B \in \mathcal{F}\}\right) \subseteq \cup\{\mathrm{Bd}B\colon B \in \mathcal{F}\}$.

IV.6. Definition. A collection $\mathcal{B}$ of open sets of X is said to be *boundary nice* if (a) for all U and V in $\mathcal{B}$, $\mathrm{Bd}\,U \cap \mathrm{Bd}\,V \cap \left(\mathrm{Bd}(U-V) \cup \mathrm{Bd}(V-U)\right) = \emptyset$ and (b) for all U in $\mathcal{B}$, $\dim(\mathrm{Bd}\,U) \leq 0$.

IV.7. Proposition. Let $\mathcal{B}$ be a boundary nice collection of open sets of X. Then

(1) for all U and V in $\mathcal{B}$, $\mathrm{Bd}\,U \cap V$ is closed

(2) for all U and V in $\mathcal{B}$, $\mathrm{C}\ell(U-V) \cap \mathrm{C}\ell(V-U) = \emptyset$

(3) $[\mathcal{B}]$ is boundary nice.

Proof: (1) $\mathrm{C}\ell(\mathrm{Bd}\,U \cap V) - V \subseteq \mathrm{Bd}\,U \cap \mathrm{Bd}\,V \cap \mathrm{Bd}(V-U) = \emptyset$. Hence $\mathrm{C}\ell(\mathrm{Bd}\,U \cap V) = \mathrm{Bd}\,U \cap V$.

(2) $\mathrm{C}\ell(U-V) \cap \mathrm{C}\ell(V-U) \subseteq \mathrm{Bd}\,U \cap \mathrm{Bd}\,V \cap \mathrm{Bd}(U-V) = \emptyset$.

(3) Let $\mathcal{F}$ and $\mathcal{G}$ be two finite subcollections of a boundary nice collection of open sets. It suffices to show that $\mathrm{Bd}(\cap\mathcal{F}) \cap \mathrm{Bd}(\cap\mathcal{G}) \cap \mathrm{Bd}(\cap\mathcal{F} - \cap\mathcal{G}) = \emptyset$ and $\mathrm{Bd}(\cup\mathcal{F}) \cap \mathrm{Bd}(\cup\mathcal{G}) \cap \mathrm{Bd}(\cup\mathcal{F} - \cup\mathcal{G}) = \emptyset$. Let us tackle the former equation.

$$\mathrm{Bd}(\cap\mathcal{F}) \cap \mathrm{Bd}(\cap\mathcal{G}) \cap \mathrm{Bd}(\cap\mathcal{F}-\cap\mathcal{G}) = \mathrm{Bd}(\cap\mathcal{F}) \cap \mathrm{Bd}(\cap\mathcal{G}) \cap \mathrm{C}\ell(\cap\mathcal{F}-\cap\mathcal{G})$$
$$\subseteq \bigcup_{F\in\mathcal{F}} \mathrm{Bd}\,F \cap \bigcap_{G\in\mathcal{G}} \mathrm{C}\ell\,G \cap \bigcup_{G\in\mathcal{G}} \mathrm{C}\ell\left(\bigcap_{F\in\mathcal{F}} (F-G)\right).$$

Thus if $p \in$ L.H.S. then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that

$p \in \mathrm{Bd}\,F \cap C\ell\,G \cap C\ell(F-G) \subseteq \mathrm{Bd}\,F \cap \mathrm{Bd}\,G \cap \mathrm{Bd}(F-G) = \emptyset$. For the latter equation we have

$$\mathrm{Bd}(\cup\mathcal{F}) \cap \mathrm{Bd}(\cup\mathcal{G}) \cap \mathrm{Bd}(\cup\mathcal{F}-\cup\mathcal{G}) = \mathrm{Bd}(\cup\mathcal{F}) \cap \mathrm{Bd}(\cup\mathcal{G}) \cap C\ell(\cup\mathcal{F}-\cup\mathcal{G})$$
$$\subseteq (X-\cup\mathcal{F}) \cap \bigcup_{G\in\mathcal{G}} \mathrm{Bd}\,G \cap \bigcup_{F\in\mathcal{F}} \Big(C\ell\Big(\bigcap_{G\in\mathcal{G}} (F-G)\Big)\Big).$$

Thus if $p \in$ L.H.S. then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that

$$p \in (X-F) \cap \mathrm{Bd}\,G \cap C\ell(F-G) \subseteq \mathrm{Bd}\,F \cap \mathrm{Bd}\,G \cap \mathrm{Bd}(F-G) = \emptyset.$$

Furthermore, from the equations $\mathrm{Bd}(U \cup V) \subseteq \mathrm{Bd}\,U \cup \mathrm{Bd}\,V$ and $\mathrm{Bd}(U \cap V) \subseteq \mathrm{Bd}\,U \cup \mathrm{Bd}\,V$ we conclude that for all $W \in [\mathcal{B}]$, $\dim(\mathrm{Bd}\,W) \leq 0$. □

We quote the following theorem established by K. Morita [25]. A proof can also be found in A.R. Pears [27].

Let Y be a regular space. The following are equivalent:

1. Y is a metric space with $\dim Y \leq n$.
2. There exists a σ-locally finite base $\mathcal{U}$ such that

   a) for any $n+1$ elements $U_1,\cdots,U_n$ of $\mathcal{U}$, $\bigcap_{i=1}^{n+1} (\mathrm{Bd}\,U_i) = \emptyset$

   b) for any $U \in \mathcal{U}$, $\dim(\mathrm{Bd}\,U) \leq n-1$.

A *base ring* $\mathcal{B}$ for a space is a base for the open sets such that $\mathcal{B} = [\mathcal{B}]$.

IV.8. Lemma. X has a countable base ring $\mathcal{B}$ which is boundary nice.

Proof: Applied to our situation, the aforementioned theorem of Morita's supplies a countable base $\mathcal{B}'$ such that for distinct U and V in $\mathcal{B}'$, $\mathrm{Bd}\,U \cap \mathrm{Bd}\,V = \emptyset$ and for all $U \in \mathcal{B}'$, $\dim(\mathrm{Bd}\,U) \leq 0$. Such a $\mathcal{B}'$ is clearly boundary nice. Let $\mathcal{B} = [\mathcal{B}']$. By Proposition IV.7(3), $\mathcal{B}$ is

our required base. □

The property of having disjoint boundaries does not carry over to the ring generated; it was partly for this reason that we introduced the idea of boundary nice.

The following proof is similar with that in Hurewicz and Wallman [14] (pg. 177). However, to produce an open set in the given base ring $\mathcal{B}$, the extra hypotheses of compactness and A being closed are needed. For this reason we include the proof.

IV.9. Lemma. Let X be a compact metric space with $\dim X \leq 1$. Let $\mathcal{B}$ be a countable base ring for X such that for all $U \in \mathcal{B}$, $\dim(\mathrm{Bd}\, U) \leq 0$. Let C_1 and C_2 be disjoint closed sets of X. Let A be a closed set in X with $\dim(A) \leq 0$. Then there exists $B \in \mathcal{B}$ with $C_1 \subseteq B$, $C\ell\, B \cap C_2 = \emptyset$ and $\mathrm{Bd}\, B \cap A = \emptyset$.

Proof: Since $\mathcal{B}$ is a base ring and X is compact there exist U and V in $\mathcal{B}$ with $C_1 \subseteq U \subseteq C\ell\, U \subseteq V \subseteq C\ell\, V \subseteq X - C_2$. Since $C\ell(U) \cap A$ is closed in A, $V \cap A$ is open in A and $C\ell\, U \cap A \subseteq V \cap A$ there exists A' open in A with $C\ell\, U \cap A \subseteq A' \subseteq V \cap A$ and $\mathrm{Bd}_A A' = \emptyset$ i.e. A' is also closed in A hence closed in X. $C_1 \cup A'$ and $C\ell(A-A')$ are disjoint closed sets. For, $A - A'$ is closed in A, therefore $C\ell(A-A') \cap A = A - A'$, thus $C\ell(A-A') \cap A' = \emptyset$. Since $A - A' \subseteq A - U$, $C\ell(A-A') \subseteq A - U$. Hence $C\ell(A-A') \cap C_1 = \emptyset$. Thus $(C_1 \cup A') \cap C\ell(A-A') = \emptyset$. Let $W \in \mathcal{B}$ such that $C_1 \cup A' \subseteq W$ and $C\ell\, W \cap (A-A') = \emptyset$. Let $B = V \cap W$. Then $B \in \mathcal{B}$, $C_1 \subseteq B$ and $C\ell\, B \cap C_2 = \emptyset$. Since $B \subseteq W$ it follows that $C\ell\, B \cap (A-A') = \emptyset$ so that $C\ell\, B \cap A \subseteq A'$. But $A' \subseteq B$ so that $C\ell\, B \cap A = A'$. Hence $\mathrm{Bd}\, B \cap A = C\ell\, B \cap (A-B) = A' - B = \emptyset$. □

Here is the main result of this chapter.

IV.10. Theorem. A compact metric space X of dimension one has breadth two.

Proof: Let $\mathcal{B}$ be a countable base ring for X which is boundary nice. Without loss of generality assume $\{\emptyset, X\} \subseteq \mathcal{B}$. $\mathcal{B}$ is then a 0-1 distributive lattice under union and intersection. We shall show that there exists $\mathcal{S} \subseteq \mathcal{B}$ with $b(\mathcal{S}) = 2$ and $\mathcal{B} = \{\cap\mathcal{F}: \mathcal{F}$ is a finite subcollection of $\mathcal{S}\}$. This $\mathcal{S}$ will be our required subbase.

Let $\mathcal{B} = \{B_i: i < \omega\}$. Assume open sets $O_1, O_2, \cdots, O_n$ have been constructed such that

1. for all $1 \leq i \leq n$, $O_i \in \mathcal{B}$
2. $b\{O_1, O_2, \cdots, O_n\} = 2$
3. $\{B_1, B_2, \cdots, B_\tau\} \subseteq \{\cap\mathcal{F}: \mathcal{F} \subseteq \{O_1, O_2, \cdots, O_n\}\}$.

Now consider $B_{\tau+1}$. Finitely many sets in $\mathcal{B}$ will be found which intersect in $B_{\tau+1}$ and which together with $\{O_1, O_2, \cdots, O_n\}$ have breadth two. To do this, Proposition II.3 is employed. That is, for each pair $O_i \neq O_j$, we find open sets O_{ij} and O_{ji} that are elements of $\mathcal{B}$ and satisfy the hypotheses of the proposition. The resulting sets that the proposition supplies will be members of $\mathcal{B}$, will intersect in $B_{\tau+1}$ and will, together with $\{O_1, O_2, \cdots, O_n\}$ have breadth two. Thus ending the proof.

Consider $\{(j,k): 1 \leq j < k \leq n\}$. Endow this set with the lexicographic ordering.

Part A. Construction of O_{kj} where $1 \leq j < k \leq n$.

Assume we have constructed O_{kj} for $(j,k) < (r,s)$ such that

A1. $O_{kj} \in \mathcal{B}$

A2. $(a,b) < (r,s)$ and $(a,b) \neq (j,k)$ implies $\mathrm{Bd}\, O_{kj} \cap C\ell\, O_j \cap \mathrm{Bd}\, O_{ba} = \emptyset$

$1 \leq \ell \leq n$ implies $\mathrm{Bd}\, O_{kj} \cap C\ell\, O_j \cap \mathrm{Bd}\, O_\ell = \emptyset$

A3. $(O_k - O_j) \cup \bigcup_{j<\ell<k} (\mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j \cap O_k) \subseteq O_{kj} \subseteq O_k$

A4. $C\ell\, O_{kj} \cap [C\ell(O_j - O_k) \cup (\mathrm{Bd}\, O_j \cap \mathrm{Bd}\, O_k) \cup \bigcup_{i<j\leq\ell\leq k} (\mathrm{Bd}\, O_{\ell i} \cap C\ell\, O_i \cap C\ell\, O_j) \cup$

$$\bigcup_{i<j} (C\ell\, O_i \cap C\ell\, O_j \cap C\ell\, O_{ji})] = \emptyset.$$

Now to define O_{sr}. First we show that $C_1 = C\ell(O_s - O_r) \cup \bigcup_{r<\ell<s} (\mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap C\ell\, O_s)$ and $C_2 = C\ell(O_r - O_s) \cup (\mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_s) \cup \bigcup_{i<r\leq\ell\leq s} (\mathrm{Bd}\, O_{\ell i} \cap C\ell\, O_i \cap C\ell\, O_r) \cup \bigcup_{i<r} (C\ell\, O_i \cap C\ell\, O_r \cap C\ell\, O_{ri})$ are disjoint closed sets. This follows from the following observations.

(a) $C\ell(O_s - O_r) \cap C\ell(O_r - O_s) = \emptyset$ (boundary nice)

(b) $C\ell(O_s - O_r) \cap \mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_s \subseteq \mathrm{Bd}(O_s - O_r) \cap \mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_s = \emptyset$

(c) $i < r \leq \ell \leq s$ implies $C\ell(O_s - O_r) \cap \mathrm{Bd}\, O_{\ell i} \cap C\ell\, O_i \cap C\ell\, O_r \subseteq \mathrm{Bd}\, O_{\ell i} \cap C\ell\, O_i \cap \mathrm{Bd}\, O_r = \emptyset$ (A2)

(d) $i < r$ implies $C\ell(O_s - O_r) \cap C\ell\, O_i \cap C\ell\, O_r \cap C\ell\, O_{ri} \subseteq \mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_i \cap C\ell\, O_{ri} = \emptyset$ (A4)

(e) $r < \ell < s$ implies $\mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap C\ell\, O_s \cap C\ell(O_r - O_s) \subseteq \mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap \mathrm{Bd}\, O_s = \emptyset$ (A2)

(f) $r < \ell < s$ implies $\mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap C\ell\, O_s \cap \mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_s \subseteq \mathrm{Bd}\, O_{\ell r} \cap \mathrm{Bd}\, O_r \cap \mathrm{Bd}\, O_\ell = \emptyset$ (A4)

(g) $r < \ell < s$ and $i < r \leq m \leq s$ implies

$\mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap C\ell\, O_s \cap \mathrm{Bd}\, O_{mi} \cap C\ell\, O_i \cap C\ell\, O_r \subseteq \mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap \mathrm{Bd}\, O_{mi} = \emptyset$ (A2)

(h) $i < r < \ell < s$ implies $\mathrm{Bd}\, O_{\ell r} \cap C\ell\, O_r \cap C\ell\, O_s \cap C\ell\, O_i \cap C\ell\, O_r \cap C\ell\, O_{ri} = \emptyset$

(A4, since $(r,\ell) < (r,s)$ and therefore $C\ell\, O_{\ell r} \cap C\ell\, O_i \cap C\ell\, O_r \cap$

$C\ell\, O_{ri} = \emptyset$).

Let $A = \bigcup_{(a,b)<(r,s)} \mathrm{Bd}\, O_{ba} \cup \bigcup_{1\leq \ell \leq n} \mathrm{Bd}\, O_\ell$. Then A is a closed subset of X with $\dim A \leq 0$. Invoking Lemma IV.9, there exists $V_{sr} \in \mathcal{B}$ with $C_1 \subseteq V_{sr}$, $C\ell\, V_{sr} \cap C_2 = \emptyset$ and $\mathrm{Bd}\, V_{sr} \cap A = \emptyset$. Let $O_{sr} = V_{sr} \cap O_s$. Then O_{sr} satisfies A1, A3 and A4. To establish A2 we prove the following: $\mathrm{Bd}\, O_{sr} \cap C\ell\, O_r \subseteq \mathrm{Bd}\, V_{sr}$.

$$
\begin{aligned}
\mathrm{Bd}\, O_{sr} \cap C\ell\, O_r &= \mathrm{Bd}(V_{sr} \cap O_s) \cap C\ell\, O_r \cap C\ell\, V_{sr} \\
&\subseteq (\mathrm{Bd}\, V_{sr} \cup \mathrm{Bd}\, O_s) \cap C\ell\, O_r \cap C\ell\, V_{sr} \\
&\subseteq \mathrm{Bd}\, V_{sr} \cup (\mathrm{Bd}\, O_s \cap C\ell\, O_r \cap C\ell\, V_{sr}) \\
&= \mathrm{Bd}\, V_{sr} \cup (\mathrm{Bd}\, O_s \cap \mathrm{Bd}\, O_r \cap C\ell\, V_{sr}) \quad (C\ell\, V_{sr} \cap (O_r - O_s) = \emptyset) \\
&= \mathrm{Bd}\, V_{sr}\,.
\end{aligned}
$$

This completes the construction of O_{sr}.

<u>Part B</u>. <u>Construction of O_{jk} where</u> $1 \leq j < k \leq n$.

For each $1 \leq j < k \leq n$, we would like to replace $\mathrm{Bd}\, O_{kj} \cap C\ell\, O_j$ by a larger open set $U_{kj} \subseteq O_k$ and still have A3 and A4. To do this realize that for $j < \ell$, $\mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j \subseteq O_\ell$. For $(\mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j) - O_\ell \subseteq \mathrm{Bd}\, O_{\ell j} \cap \mathrm{Bd}\, O_\ell \cap \mathrm{Bd}\, O_j = \emptyset$. Now, for $j < \ell < k$, $\mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j \cap C\ell(O_k - O_{kj}) \subseteq \mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j \cap \mathrm{Bd}\, O_k = \emptyset$ (A3, A2). Hence we can find $V^k_{\ell j} \in \mathcal{B}$ such that $\mathrm{Bd}\, O_{\ell j} \cap C\ell\, O_j \subseteq V^k_{\ell j} \subseteq O_\ell$ and $V^k_{\ell j} \cap C\ell(O_k - O_{kj}) = \emptyset$ which implies $V^k_{\ell j} \cap O_k \subseteq O_{kj}$. It now follows that by using normality, the fact that $\mathcal{B}$ is a base ring for a compact

space and by repeatedly intersecting the open sets created we can construct by lexicographic induction open sets U_{kj} for $1 \le j < k \le n$ such that the following holds:

B1. O_{kj} and U_{kj} are members of $\mathcal{B}$.

B2. $\mathrm{Bd}\, O_{kj} \cap \mathrm{C\ell}\, O_j \subseteq U_{kj} \subseteq O_k$.

B3. $(O_k - O_j) \cup \bigcup_{j<\ell<k} (U_{\ell j} \cap O_k) \subseteq O_{kj} \subseteq O_k$.

B4. $(\mathrm{C\ell}\, O_{kj} \cup \mathrm{C\ell}\, U_{kj}) \cap [\mathrm{C\ell}(O_j - O_k) \cup (\mathrm{Bd}\, O_j \cap \mathrm{Bd}\, O_k) \cup \bigcup_{i<j\le\ell\le k} (U_{\ell i} \cap \mathrm{C\ell}\, O_j) \cup \bigcup_{i<j} (\mathrm{C\ell}\, O_i \cap \mathrm{C\ell}\, O_j \cap \mathrm{C\ell}\, O_{ji})] = \emptyset$.

The stage is now set to define O_{jk}. Denote the longer expression in B4 by R. Since $\mathrm{C\ell}\, O_{kj} \subseteq O_{kj} \cup U_{kj} \cup (X - \mathrm{C\ell}\, O_j)$ we know that $\mathrm{C\ell}\, O_{kj} \cap [\mathrm{C\ell}\, O_j \cap X - (O_{kj} \cup U_{kj})] = \emptyset$. Let $V_{jk} \in \mathcal{B}$ such that $R \subseteq \mathrm{C\ell}\, O_j \cap X - (O_{kj} \cup U_{kj}) \subseteq V_{jk}$ and $\mathrm{C\ell}\, V_{jk} \cap \mathrm{C\ell}\, O_{kj} = \emptyset$. Let $O_{jk} = V_{jk} \cap O_j$. Thus the following holds:

C1. O_{kj}, U_{kj} and O_{jk} are members of $\mathcal{B}$.

C2. $\mathrm{C\ell}\, O_{jk} \cap \mathrm{C\ell}\, O_{kj} = \emptyset$.

C3. $O_{kj} \cup U_{kj} \cup O_{jk} = O_j \cup O_k$.

C4. $(O_k - O_j) \cup \bigcup_{j<\ell<k} (U_{\ell j} \cap O_k) \subseteq O_{kj} \subseteq O_k$

C5. $(O_j - O_k) \cup \bigcup_{i<j\le\ell\le k} (U_{\ell i} \cap O_j) \cup \bigcup_{i<j} (O_i \cap O_j \cap O_{ji}) \subseteq O_{jk} \subseteq O_j$.

<u>Part C</u>. <u>The hypotheses of Proposition II.3 are satisfied by</u> O_{jk} <u>and</u> O_{kj}

Hypothesis (1). C4 and C5 imply $O_{jk} \subseteq O_j$, $O_{kj} \subseteq O_k$ and

$$O_{jk} \cup O_k = O_{kj} \cup O_j = O_j \cup O_k.$$

Hypothesis (2). C2 implies $O_{jk} \cap O_{kj} = \emptyset$

Hypothesis (3). Let $i < j < k$

(a) C5 yields $O_i \cap O_j \cap O_{ji} \subseteq O_{jk}$ and also with $\ell = j$, $U_{ji} \cap O_j \subseteq O_{jk}$. The latter, using C3, yields $(O_i \cup O_j) \cap O_j \subseteq O_{jk} \cup O_{ij} \cup O_{ji}$. Therefore,

$$\begin{aligned} O_i &= (O_i - O_j) \cup (O_i \cap O_j) \\ &\subseteq O_{ij} \cup [(O_i \cap O_j \cap O_{ji}) \cup (O_i \cap O_j \cap U_{ji}) \cup (O_i \cap O_j \cap O_{ij})] \\ &\subseteq O_{ij} \cup O_{jk}. \end{aligned}$$

(b) C5 with $\ell = k$ yields $U_{ki} \cap O_j \subseteq O_{jk}$. Using C3, we get $(O_i \cup O_k) \cap O_j \subseteq O_{jk} \cup O_{ik} \cup O_{ki}$. Therefore,

$$\begin{aligned} O_j &= (O_j - O_k) \cup (O_k \cap O_j) \\ &\subseteq O_{jk} \cup O_{ik} \cup O_{ki}. \end{aligned}$$

(c) C4 yields $U_{ji} \cap O_k \subseteq O_{ki}$. Using C3 we get $(O_i \cup O_j) \cap O_k \subseteq O_{ki} \cup O_{ij} \cup O_{ji}$. Therefore,

$$\begin{aligned} O_k &= (O_k - O_i) \cup (O_i \cap O_k) \\ &\subseteq O_{ki} \cup O_{ij} \cup O_{ji}. \end{aligned}$$

Ignoring the order, we surely have: For distinct i,j,k,

$$O_i \subseteq O_{jk} \cup O_{kj} \cup O_{ij} \cup O_{ik}.$$

Hypothesis (4a). We did this in part (a) above.

Hypothesis (4b). Let $i < j < k < \ell$. C5 yields $U_{ki} \cap O_j \subseteq O_{j\ell}$. Using C3, we get $(O_i \cup O_k) \cap O_j \subseteq O_{j\ell} \cup O_{ik} \cup O_{ki}$. Therefore,

$$\begin{aligned} O_k &= (O_k - O_j) \cup (O_k \cap O_j) \\ &\subseteq O_{kj} \cup O_{j\ell} \cup O_{ik} \cup O_{ki}. \end{aligned}$$

This concludes the proof. $\square$

IV.11. Corollary. A separable metric space of dimension one has breadth two.

Proof: K. Menger [18] has constructed a universal one-dimensional separable metric space which is also compact. The result follows from the monotonicity of the breadth function. □

The techniques used in these results seem to be peculiarly one-dimensional in nature. Since all non-degenerate 0-dimensional and 1-dimensional separable metric spaces have breadth two, breadth does not depend upon dimension. The following intriguing question arises. Do all separable metric spaces have breadth two? Equivalently, is the breadth of the Hilbert cube two?

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